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A coupled channel model of scattering with $SO(3,1)$ symmetry

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Abstract
An exactly solvable coupled channel scattering problem with $SO(3,1)$ symmetry is presented describing the helicity scattering of a particle with spin s. It is shown that the coupled channel wavefunction is a matrix-valued function with definite group theoretical properties. The scattering phase shifts are calculated for the special values of $s = \frac{1}{2}$, 1 and $\frac{3}{2}$ and the result for general s is conjectured. It is also demonstrated that for an algebraic description of this coupled channel problem both of the independent Casimir operators are needed.

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1. Introduction

The importance of symmetry principles embodied in group theoretical methods in theoretical physics is well known. The spectacular success of symmetry groups, and their attendant algebras in high-energy and elementary particle physics are the most obvious examples. However, it is by no means such a common wisdom that spectrum generating groups and algebras also are useful in the description of low-energy processes. Such group theoretic methods were applied to bound-state problems [1] (of molecular and nuclear systems) in the first instance. However, only after the advent of algebraic scattering theory (AST) did such methods become relevant in studies of the scattering regime. AST successfully described nonrelativistic scattering problems of a wide range by using *noncompact* symmetry groups [2] and the method was generalized in principle to include coupled channel problems [3]; however, some conceptual problems remained and such form the raison d'etre of this paper.

AST is a purely group theoretic method to specify the scattering matrix in the sense that only the noncompact symmetry group G and its subgroup structure characterizing the scattering process are needed as input. No explicit coordinate realizations of interaction terms and channel potentials appear in this approach. The only assumption is that the scattering system is described by a Hamiltonian describable as a function of the quadratic Casimir operator of G. With such a Hamiltonian the theory of group contractions and expansions facilitates an explicit algebraic determination of the functional form of the scattering matrix.

However, with higher-rank groups there are diverse Casimir operators and the dynamical role of such has been completely neglected in AST. In AST it has been assumed that the scattering states are described merely by those unitary irreducible representations of G for which the eigenvalues of these extra Casimir operators are zero. This restriction could be deleterious as far as physical applications are concerned. Heuristically one can argue that the additional labels provided by these extra Casimir operators might be used effectively to label possible scattering channels in an intrinsically algebraic manner. The possible role of the extra Casimir operators in this spirit has been emphasized in [4], suggesting an algebraic characterization of some scattering channels for a multichannel process. But those studies, contrary to the spirit of AST, employed an explicit coordinate realization, yielding explicit interaction terms. In this presentation, we consider an explicitly solvable model to clearly show the importance of abstract mathematical issues. To the best of our knowledge, in the literature, no exactly solvable group theoretical model of this kind has appeared. Specifically we introduce and solve a multichannel scattering model having $SO(3,1)$ symmetry. We show that for this model expectations that the extra Casimir operators in fact provide essential new labels characterizing the channel structure of the scattering problem are fulfilled. Our model describes scattering of a particle having an intrinsic spin s in a helicity formalism.

In section 2 we introduce our particular form of realization for the $SO(3,1)$ algebra providing the infinitesimal generators of the group $SO(3,1)$ with which there are *two* independent Casimir operators. One of those Casimirs has the form of a 'Schrödinger-like' operator; the other is a 'Dirac-like' operator. In section 3 we study the asymptotic behaviour of the scattering states characterized as eigenstates of these Casimir operators. Therein we also identify the observables of the scattering process. The short-distance behaviour is investigated in section 4, where we show that the dynamics in this limit is just the free dynamics in $\boldsymbol{R}^3$. The relevant symmetry group is the Euclidean group $E(3)$ arising as a contraction of $SO(3,1)$. Detailed discussion of the coupled channel problem is given in section 5, while the explicit solution of the eigenvalue problem for the Casimir operators in terms of known special functions is presented in section 6. The special cases of spin $\frac{1}{2}$, 1 and $\frac{3}{2}$ are studied in detail in separate subsections. The asymptotic behaviour of the coupled channel wavefunction is discussed in section 7. The scattering matrix is explicitly calculated for the cases $s = \frac{1}{2}$, 1 and $\frac{3}{2}$ in section 8. Here the general form of the scattering matrix is also conjectured. The conclusions are left for section 9. For the convenience of the reader we also have included two appendices. In the appendix we explicitly check that the states obtained for the spin 1 case are also eigenstates of the first-order Casimir operator.

2. A matrix-valued realization for $SO(3,1)$

Defining coordinates x^μ, $\mu = 0, 1, 2, 3$ on the upper sheet of the double-sheeted hyperboloid defined by

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 1 \qquad x^0 \geqslant 1 \tag{1}$$

then $SO(3,1)$ (the proper orthochronous Lorentz group), acting on the coordinates as $x^\mu \to \Lambda^\mu_\nu x^\nu$ with $\Lambda^\mu_\nu \in SO(3,1)$, leaves invariant this hyperboloid. The infinitesimal generators of this action,

$$L_j = -\mathrm{i}\epsilon_{jkl}x^k \frac{\partial}{\partial x^l} \qquad K_j = -\mathrm{i}\left(x^0 \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial x^0}\right) \tag{2}$$

satisfy the commutation relations of the $SO(3,1)$ algebra,

$$\begin{aligned} &[L_j, L_m] = \mathrm{i}\epsilon_{jmn} L_n \qquad [L_j, K_m] = \mathrm{i}\epsilon_{jmn} K_n \\ &[K_j, K_m] = -\mathrm{i}\epsilon_{jmn} L \qquad j, m, n = 1, 2, 3. \end{aligned} \tag{3}$$

The generators $\boldsymbol{L}$ and $\boldsymbol{K}$ can be thought of as the ones generating infinitesimal rotations or hyperbolic rotations (Lorentz transformations) respectively.

According to the results of [5] this realization of the $SO(3,1)$ algebra can be further generalized by adding suitable matrix-valued modifications to the generators $\boldsymbol{L}$ and $\boldsymbol{K}$. Let us define $\boldsymbol{S}$ as the usual spin matrices representing a particle with spin s. They are $(2s+1)\times(2s+1)$ matrices satisfying the commutation relations $[S_j, S_k] = \mathrm{i}\epsilon_{jkl} S_l$. Then it can be shown [5] that the modified generators

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \qquad \boldsymbol{M} = \boldsymbol{K} + \frac{1}{1+x^0}\boldsymbol{S}\times\boldsymbol{x} \tag{4}$$

satisfy the same set of commutation relations i.e.

$$\begin{aligned} &[J_j, J_m] = \mathrm{i}\epsilon_{jmn} J_n \qquad [J_j, M_m] = \mathrm{i}\epsilon_{jmn} M_n \\ &[M_j, M_m] = -\mathrm{i}\epsilon_{jmn} J_n \qquad j, m, n = 1, 2, 3. \end{aligned} \tag{5}$$

Notice that the generators given in equation (4) are matrix-valued differential operators; the geometric meaning of such generators was explained in [6]. There an explicit construction was given for a particular choice of a semisimple Lie group G, and any subgroup H rendering the coset G/H a symmetric space. Choosing an irreducible unitary representation $\mathcal{D}$ for H, and local coordinates for G/H, it was shown that the generators of the representation of G induced by $\mathcal{D}$ are matrix-valued differential operators of the equation (5) form. Indeed, in this case $G = SO(3,1)$, $H = SO(3)$, $\mathcal{D}$ is the usual spin s representation, G/H is the upper sheet of the double-sheeted hyperboloid.

Now we introduce polar coordinates

$$\boldsymbol{x} = \boldsymbol{n}\sinh r \qquad x^0 = \cosh r \qquad \boldsymbol{n}^2 = 1 \tag{6}$$

where

$$\boldsymbol{n}(\theta,\varphi) \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{7}$$

In terms of these new variables

$$\boldsymbol{M} = -\mathrm{i}\boldsymbol{n}\frac{\mathrm{d}}{\mathrm{d}r} - \coth r\,\boldsymbol{n}\times\boldsymbol{J} + \frac{\boldsymbol{n}\times\boldsymbol{S}}{\sinh r}. \tag{8}$$

The components of $\boldsymbol{L}$ in terms of θ and φ are the usual ones well known from the literature.

It is important to stress however that we can have another way of looking at our realization [5]. Let us regard our coordinates $x^\mu \equiv (x^0, \boldsymbol{x})$ as operators satisfying the constraint $x_\mu x^\mu = 1$ (indices are raised and lovered by the metric $g_{\mu\nu} \equiv \mathrm{diag}(1,-1,-1,-1)$). Then the ten generators x_μ, $N_{\mu\nu}$ ($\mu,\nu = 0,1,2,3$) where

$$N^{ij} = \varepsilon_{ijk} J_k \qquad N^{0k} = -N^{k0} = M_k \tag{9}$$

satisfy the commutation relations of the Poincaré algebra,

$$[x_\mu, x_\nu] = 0 \qquad [N_{\mu\nu}, x_\sigma] = \mathrm{i}(x_\mu g_{\nu\sigma} - x_\nu g_{\mu\sigma}) \tag{10}$$

$$[N_{\mu\nu}, N_{\rho\sigma}] = \mathrm{i}(g_{\mu\sigma}N_{\nu\rho} + g_{\nu\rho}N_{\mu\sigma} - g_{\mu\rho}N_{\nu\sigma} - g_{\nu\sigma}N_{\mu\rho}). \tag{11}$$

The irreducible unitary representations of the Poincaré algebra are labelled by the eigenvalues of the operators $x_\mu x^\mu$ (momentum squared), and $W^2 = W_\mu W^\mu$ of $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}N^{\nu\sigma}X^\rho$ (the Pauli–Lubanski operator) related to the spin. A calculation shows that $W_\mu = (W_0, \boldsymbol{W}) = (x_0\boldsymbol{J} + \boldsymbol{M}\times\boldsymbol{x}, \boldsymbol{J}\boldsymbol{x})$. Using the (4) form of the generators one can show that

$$W_0 = \boldsymbol{S}\boldsymbol{x} \qquad \boldsymbol{W} = (x_0 - 1)\boldsymbol{n}(\boldsymbol{S}\boldsymbol{n}) + \boldsymbol{S}. \tag{12}$$

Using this we get $W^2 = -S^2$, and $W_\mu x^\nu = 0$ as it has to be. Notice also that $W(x^\star) = (0, \boldsymbol{S})$, where $x^{\star\mu} \equiv (1, 0, 0, 0)$. According to the representation theory of the Poincaré group (in an irrep we have $W^2 = -m^2 s(s+1)I$, and in our case the mass squared $m^2 \equiv x_\mu x^\mu = 1$), we are given a series of unitary representations labelled as $(1, s)$, $s = 0, \frac{1}{2}, 1, \ldots$. Hence our matrix-valued realization is a natural one for introducing spin degrees of freedom.

Let us now continue by transforming our realization to a form more suitable for our purposes. First recall that $\boldsymbol{J} \times \boldsymbol{n} + \boldsymbol{n} \times \boldsymbol{J} = 2\mathrm{i}\boldsymbol{n}$ (i.e. $\boldsymbol{n}$ transforms as a vector operator with respect to the rotation subgroup $SO(3)$ of $SO(3,1)$). Using this and the similarity transformation

$$\boldsymbol{M} \to \boldsymbol{M}' \equiv \sinh r \boldsymbol{M} \frac{1}{\sinh r} \tag{13}$$

transforms equation (8) to the form

$$\boldsymbol{J}' = \boldsymbol{L} + \boldsymbol{S} \qquad \boldsymbol{M}' = -\mathrm{i}\boldsymbol{n}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{2}\coth r(\boldsymbol{n} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{n}) + \frac{\boldsymbol{n} \times \boldsymbol{S}}{\sinh r}. \tag{14}$$

(For notational simplicity in the following we drop the prime from $\boldsymbol{J}$ and $\boldsymbol{M}$.) Notice also that the similarity transformation equation (13) is just the one needed to transform the measure $\sinh^2 r \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r$ on the hyperboloid to the measure $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r$. Equation (14) is a realization in terms of matrix-valued differential operators expressed in terms of the spherical polar coordinates (r, θ, φ). Moreover, the radial coordinate is contained only in $\boldsymbol{M}$. Of course the components of the operators given in equation (14) satisfy the commutation relations equation (5). Moreover, the generators $\boldsymbol{J}$, $\boldsymbol{M}$ are Hermitian operators with respect to the scalar product, with the measure $\mathrm{d}\mu = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r$.

Since $SO(3,1)$ is a group of rank two, we have two independent Casimir operators. They are $\mathcal{C}_1 = \boldsymbol{J}^2 - \boldsymbol{M}^2$ and $\mathcal{C}_2 = \boldsymbol{J}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{J}$. A straightforward calculation for $\mathcal{C}_1$ yields the result [5]

$$\mathcal{C}_1 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} - 1 + (\boldsymbol{S}\boldsymbol{n})^2 + \frac{\boldsymbol{L}^2 + 4(\boldsymbol{S}\boldsymbol{J} - (\boldsymbol{S}\boldsymbol{n})^2)}{4\cosh^2 r/2} - \frac{\boldsymbol{L}^2}{4\sinh^2 r/2}. \tag{15}$$

This form of the quadratic Casimir was used [5] to investigate just the spin $\frac{1}{2}$ case. Therein we seek to gain some insight into the higher-spin cases as well, hence we try to rewrite it in a more instructive form. To do so it is useful to introduce a new set of Hermitian operators,

$$\mathcal{A} = \boldsymbol{J}^2 - (\boldsymbol{S}\boldsymbol{n})^2 \qquad \mathcal{B} = \boldsymbol{S}^2 - (\boldsymbol{S}\boldsymbol{n})^2 \tag{16}$$

and

$$\Sigma_1 = \boldsymbol{J}\boldsymbol{S} - (\boldsymbol{S}\boldsymbol{n})^2 \qquad \Sigma_2 = (\boldsymbol{S} \times \boldsymbol{n})\boldsymbol{J} \tag{17}$$

which satisfy commutation relations

$$[\boldsymbol{S}\boldsymbol{n}, \Sigma_1] = \mathrm{i}\Sigma_2 \qquad [\Sigma_2, \boldsymbol{S}\boldsymbol{n}] = \mathrm{i}\Sigma_1 \qquad [\Sigma_1, \Sigma_2] = (\mathcal{A} + \mathcal{B})\boldsymbol{S}\boldsymbol{n} \tag{18}$$

and $[\mathcal{A}, \boldsymbol{S}\boldsymbol{n}] = [\mathcal{B}, \boldsymbol{S}\boldsymbol{n}] = 0$. One can also show that all these operators are Hermitian. The commutation relations of equation (18) can be rewritten alternatively as

$$[\boldsymbol{S}\boldsymbol{n}, \Sigma_\pm] = \pm\Sigma_\pm \qquad [\Sigma_+, \Sigma_-] = 2(\mathcal{A} + \mathcal{B})\boldsymbol{S}\boldsymbol{n} \tag{19}$$

by introducing the operators

$$\Sigma_\pm = \Sigma_1 \pm \mathrm{i}\Sigma_2 \qquad {\Sigma_\pm}^\dagger = \Sigma_\mp \tag{20}$$

which act as step operators for the eigenvalue of the operator $\boldsymbol{S}\boldsymbol{n}$. In fact the algebra given by equation (19), apart from the presence of the term $\mathcal{A} + \mathcal{B}$ in one of the commutators, resembles a usual $su(2)$ algebra. However, although $\mathcal{A} + \mathcal{B}$ commutes with $\boldsymbol{S}\boldsymbol{n}$, it does not commute with $\Sigma_\pm$, so that this analogy can be taken no further.

In terms of our newly introduced quantities, the Casimir operators take the form

$$\mathcal{C}_1 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} - 1 + (\boldsymbol{S}\boldsymbol{n})^2 - \frac{\mathcal{A}+\mathcal{B}}{\sinh^2 r} + \frac{2\Sigma_1 \cosh r}{\sinh^2 r} \tag{21}$$

$$\mathcal{C}_2 = \boldsymbol{S}\boldsymbol{n}\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{\Sigma_2}{\sinh r}. \tag{22}$$

In the following sections we search for a group theoretical description of the eigenfunctions of the operators of equations (21) and (22) amenable for a description of some *nonrelativistic* scattering problem involving a scattered particle having an intrinsic spin s. In particular we would like to obtain from the group theoretical information the explicit form of coupled channel interaction terms, the coupled channel wavefunctions and the elements of the scattering matrix. It is should be clear by now that this task can also be rephrased in terms of the representation theory of the Poincaré group. Although the Poincaré group is the right group theoretical tool for deriving covariant forms of *relativistically* covariant equations, here we merely look at it as a mathematical means for describing exactly solvable *nonrelativistic* coupled channel scattering problems. We will not pursue here the interesting possibility for constructing a mathematical mapping between the relativistically covariant equations with mass equal to one and arbitrary spin (the generalized Bargmann–Wigner equations) and our equations describing multichannel scattering processes.

3. Asymptotic forms of the Casimir operators

We investigate the asymptotic behaviour of our Casimir operators $\mathcal{C}_1$ and $\mathcal{C}_2$. For this purpose we take the limit $r \to \infty$ to get

$$\mathcal{C}_1{}^{\infty} = \lim_{r\to\infty} \mathcal{C}_1 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + (\boldsymbol{S}\boldsymbol{n})^2 - 1 \qquad \mathcal{C}_2{}^{\infty} = \lim_{r\to\infty} \mathcal{C}_2 = \boldsymbol{S}\boldsymbol{n}\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r}\right). \tag{23}$$

These operators, and others commuting with them, describe the physical situation asymptotically. First notice that the operators $\boldsymbol{J}$, $\boldsymbol{S}^2$ and $\boldsymbol{S}\boldsymbol{n}$ are mutually commuting, and they also commute with the Casimir operators given in equation (23). Hence to give the asymptotic description of the scattering states, we can use the commuting set of operators $(\mathcal{C}_1{}^{\infty}, \mathcal{C}_2{}^{\infty}, \boldsymbol{J}^2, J_3, \boldsymbol{S}^2, \boldsymbol{S}\boldsymbol{n})$. In the following we describe the scattering situation with quantum numbers corresponding to eigenvalues of these observables.

As a first step recall [5] that scattering states of a system having $SO(3,1)$ symmetry can be labelled by the *pair* of quantum numbers (j_0, j_1) where j_0 is purely *imaginary* and $j_1 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Choosing a particular pair (j_0, j_1) from the above set amounts to labelling the corresponding scattering state by a *unitary irreducible representation* of the algebra $SO(3,1)$. Moreover, the numbers (j_0, j_1) are related to the eigenvalues of the Casimir operators as follows [7]:

$$\mathcal{C}_1|j_0 j_1\rangle = (j_0{}^2 + j_1{}^2 - 1)|j_0 j_1\rangle \qquad \mathcal{C}_2|j_0 j_1\rangle = -\mathrm{i} j_0 j_1 |j_0 j_1\rangle. \tag{24}$$

Of course an identical pair of equations should hold also for the asymptotic operators with the asymptotic states $|j_0 j_1; jms\lambda\rangle^{\infty}$,

$$\mathcal{C}_1{}^{\infty}|j_0 j_1; jms\lambda\rangle^{\infty} = (j_0{}^2 + j_1{}^2 - 1)|j_0 j_1; jms\lambda\rangle^{\infty} \tag{25}$$

$$\mathcal{C}_2{}^{\infty}|j_0 j_1; jms\lambda\rangle^{\infty} = -\mathrm{i} j_0 j_1 |j_0 j_1; jms\lambda\rangle^{\infty} \tag{26}$$

where the extra labels are ones corresponding to the remaining operators in the set $(\mathcal{C}_1{}^{\infty}, \mathcal{C}_2{}^{\infty}, \boldsymbol{J}^2, J_3, \boldsymbol{S}^2, \boldsymbol{S}\boldsymbol{n})$. This means that we have also the equations

$$\boldsymbol{J}^2|j_0 j_1; jms\lambda\rangle^{\infty} = j(j+1)|j_0 j_1; jms\lambda\rangle^{\infty} \tag{27}$$

$$J_3|j_0 j_1; jms\lambda\rangle^{\infty} = m|j_0 j_1; jms\lambda\rangle^{\infty} \tag{28}$$

and

$$\boldsymbol{S}^2|j_0j_1;\, jms\lambda\rangle^\infty = s(s+1)|j_0j_1;\, jms\lambda\rangle^\infty \tag{29}$$

$$\boldsymbol{Sn}|j_0j_1;\, jms\lambda\rangle^\infty = \lambda|j_0j_1;\, jms\lambda\rangle^\infty \tag{30}$$

where $-s \leqslant \lambda \leqslant s$. The physical interpretation of the eigenvalue λ is clear. It is the *helicity* quantum number of the scattered particle with spin s.

Since we have an explicit coordinate realization, in the equations above instead of the abstract ket vectors $|\rangle^\infty$ we could use the wavefunction

$$\Psi^\infty_{j_0j_1;jms\lambda}(r,\theta,\varphi) \equiv \langle r,\theta,\varphi|j_0j_1;\, jms\lambda\rangle^\infty. \tag{31}$$

The operators in these equations are $(2s+1)\times(2s+1)$ matrix-valued differential operators. The matrices can be simultaneously diagonalized, by the transform that diagonalizes the matrix $\boldsymbol{Sn}$. That is achieved by using the unitary matrix

$$U(\theta,\varphi) = \mathrm{e}^{-\mathrm{i}\varphi S_3}\mathrm{e}^{-\mathrm{i}\theta S_2}\mathrm{e}^{\mathrm{i}\varphi S_3} \equiv D^s_{\lambda\nu}(\varphi,\theta,-\varphi) \qquad -s \leqslant \lambda \quad \nu \leqslant s \tag{32}$$

where in equation (32) the second equality reveals the connection between our matrix U and Wigner's D function for spin s. The action on $\boldsymbol{Sn}$ gives

$$U^\dagger(\theta,\varphi)\boldsymbol{Sn}U(\theta,\varphi) = S_3. \tag{33}$$

This transformation diagonalizes the matrix-valued operator $\boldsymbol{J}$ as well, giving [8, 9]

$$\boldsymbol{J}' \equiv U^\dagger \boldsymbol{J} U = \boldsymbol{r}\times(\boldsymbol{p}+\boldsymbol{A}) + S_3\boldsymbol{n} = \boldsymbol{L}+\boldsymbol{W} \tag{34}$$

where

$$\boldsymbol{A} = \frac{1}{r(r+r_3)}\begin{pmatrix}-r_2\\ r_1\\ 0\end{pmatrix}S_3 \qquad \boldsymbol{W} = \frac{1}{(r+r_3)}\begin{pmatrix}r_1\\ r_2\\ 0\end{pmatrix}S_3. \tag{35}$$

In these equations $\boldsymbol{A}$ is a diagonal matrix-valued vector potential containing $(2s+1)$ magnetic monopole vector-potentials, where the pole-strength is just the helicity eigenvalue of $\boldsymbol{Sn}$. Notice also that although in these formulae we have used the vector $\boldsymbol{r} = r\boldsymbol{n}$, and the operator $\boldsymbol{p} = -\mathrm{i}\nabla_{\boldsymbol{r}}$, the diagonal operator $\boldsymbol{J}'$ does *not* depend on r. It depends merely on $\boldsymbol{n}$ and solely on the angular variables (θ,φ).

Thus the joint eigenfunctions of $\boldsymbol{J}^2$, J_3, $\boldsymbol{S}^2$ and $\boldsymbol{Sn}$ are of the form

$$\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) = \mathcal{Y}(\theta,\varphi)U(\theta,\varphi)\chi^s_\lambda \tag{36}$$

where as usual

$$\boldsymbol{S}^2\chi^s_\lambda = s(s+1)\chi^s_\lambda \qquad S_3\chi^s_\lambda = \lambda\chi^s_\lambda. \tag{37}$$

The unknown functions $\mathcal{Y}$ satisfy the equations

$$\boldsymbol{J}'^2\mathcal{Y} = j(j+1)\mathcal{Y} \qquad J'_3\mathcal{Y} = m\mathcal{Y}. \tag{38}$$

When each component of the diagonal matrix-valued differential equations, equation (38) is written explicitly one finds defining equations of the so called monopole harmonics (see e.g. [8]), that in terms of Wigner's D-function are

$$\mathcal{Y}(\theta,\varphi) = D^j_{\lambda m}(\varphi,-\theta,-\varphi). \tag{39}$$

Note however, that the allowed values for j are restricted by the particular helicity eigenvalue λ being considered. This restriction is [8, 9]

$$j = |\lambda|, |\lambda|+1, |\lambda|+2, \ldots. \tag{40}$$

Consider now equations (25) and (26) with the explicit form for the operators and common eigenfunction given by equations (23) and (31). First by using equations (36) and (39) we can write equation (31) in the form

$$\Psi^{\infty}_{j_0 j_1; jms\lambda}(r, \theta, \varphi) \equiv \psi^{\infty}_{j_0 j_1; js\lambda}(r)\mathcal{D}^{js}_{m\lambda}(\theta, \varphi). \tag{41}$$

Then using the orthogonality of the functions $\mathcal{D}^{js}_{m\lambda}(\theta, \varphi)$, the eigenvalue problems for the functions $\psi^{\infty}_{j_0 j_1; js\lambda}(r)$ can be written in the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - {j_0}^2 + (\lambda^2 - {j_1}^2)\right)\psi^{\infty}_{j_0 j_1; js\lambda}(r) = 0 \tag{42}$$

$$\left(\lambda\frac{\mathrm{d}}{\mathrm{d}r} - j_0 j_1\right)\psi^{\infty}_{j_0 j_1; js\lambda}(r) = 0 \tag{43}$$

where $-s \leqslant \lambda \leqslant s$. By appropriately choosing the $SO(3, 1)$ representations the components of $\psi^{\infty}_{j_0 j_1; js\lambda}(r)$ can be arranged in a $2s+1$-dimensional matrix containing incoming and outgoing plane waves multiplied by suitable amplitudes depending on s and j.

To demonstrate that arrangement first choose $j_0 = \mathrm{i}k$, where $E = k^2$ is the scattering energy. Next by restricting the label j_1 by $-s \leqslant j_1 \leqslant s$, in the indices λ and j_1 $\psi^{\infty}_{j_0 j_1; js\lambda}(r)$ is a $2s + 1$-dimensional matrix. Recall that the representations $(\mathrm{i}k, j_1)$ and $(\mathrm{i}k, -j_1)$ are mirror conjugated; moreover, the representations $(-\mathrm{i}k, j_1)$ and $(\mathrm{i}k, -j_1)$ for $j_1 \neq 0$ are unitary equivalent [7]. For $j_1 = 0$ the representations $(\mathrm{i}k, 0)$ and $(-\mathrm{i}k, 0)$ are inequivalent. Hence by allowing the values of j_1 also to be negative, but k *non-negative* (as it has to be) we cover all the irreps. The case $j_1 = 0$ needs special care. But its special nature will indeed be reflected in the formalism. In the following we assume that the scattering process is described by this set of representations, which we label as

$$(j_0, j_1) = (\mathrm{i}k, \nu) \qquad k \in \boldsymbol{R}^+_0 \qquad -s \leqslant \nu \leqslant s. \tag{44}$$

In the following we use the shorthand notation

$$\psi^{\infty}_{\lambda\nu}(r) \equiv \psi^{\infty}_{k\nu; js\lambda}(r) \qquad -s \leqslant \lambda \quad \mu \leqslant s \tag{45}$$

to emphasize the matrix character of our wavefunction, the columns of which are specified by ν and belong to *different* irreducible unitary representations of $SO(3, 1)$. In accordance with the coupled channel formalism of quantum scattering theory this label can be used to specify the different boundary conditions for the wavefunction describing the scattering process. To proceed one further assumption is needed, namely that the coupled channel wave function (45) satisfies

$$\psi^{\infty}_{\lambda\nu}(r) = \psi^{\infty}_{\nu\lambda}(r) \tag{46}$$

i.e. the corresponding matrix is symmetric. The justification of this assumption is given in the next section, where the symmetry relation equation (46) will be proved for the entire $\psi_{\lambda\nu}(r)$, not merely for its asymptotic form.

Equations (42) and (43) now are expressed in the $2s + 1$-dimensional matrix form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2\right)\psi^{\infty}(r) = [\psi^{\infty}(r), S_3^2] \tag{47}$$

$$S_3\frac{\mathrm{d}}{\mathrm{d}r}\psi^{\infty}(r) = \mathrm{i}k\psi^{\infty}(r)S_3 \tag{48}$$

where the diagonal matrix S_3 selects the entries $-s \leqslant \lambda \leqslant s$ $(-s \leqslant \nu \leqslant s)$ when S_3 acts upon the matrix ψ from the left (right). Taking the matrix transpose of equation (48) and by virtue of equation (46) we get $\frac{\mathrm{d}}{\mathrm{d}r}\psi^{\infty}S_3 = \mathrm{i}kS_3\psi^{\infty}$. Multiplying this from the *left*, and equation (48) from the *right* by S_3 and then subtracting the two, gives

$$[\psi^{\infty}(r), S_3^2] = 0 \tag{49}$$

hence the term on the right-hand side of equation (47) vanishes. The solutions of equation (47) are then linear combinations of incoming and outgoing plane waves. But there is more we can show. Writing out equation (49) we get $(\lambda^2 - \nu^2)\psi^{\infty}_{\lambda\nu}(r) = 0$, which shows that the only nonvanishing components of $\psi^{\infty}_{\lambda\nu}(r)$ satisfy the constraint $\lambda = \pm\nu$. Then from equation (48) we have $\frac{\mathrm{d}}{\mathrm{d}r}\psi^{\infty} = \pm\mathrm{i}k\psi^{\infty}$. Hence for $\lambda \neq 0$

$$\psi^{\infty}_{\lambda,\pm\lambda}(r) = A^{\pm}_{js\lambda}(k)\mathrm{e}^{\pm\mathrm{i}kr} \qquad \lambda \neq 0. \tag{50}$$

For $\lambda = \nu = 0$ there is no restriction dictated by equation (48), whence one finds

$$\psi^{\infty}_{00}(r) = A^{+}_{js0}(k)\mathrm{e}^{\mathrm{i}kr} + A^{-}_{js0}(k)\mathrm{e}^{-\mathrm{i}kr} \qquad \lambda = 0. \tag{51}$$

Hence the scattering problem described by our realization is a helicity scattering process. The incident particle is one with spin s and a definite helicity with respect to its instantaneous direction of motion, and suffers a helicity flip on beeing scattered. Our task is now to find the amplitudes $A^{\pm}_{js\lambda}(k)$ and to calculate the scattering matrix. But before solving the eigenvalue problem of the Casimir operators (and from the asymptotic form determining the scattering matrix) it is useful to investigate the short-distance limit as well.

4. The short-distance behaviour of the Casimir operators

The $r \to 0$ limit for our generators $\boldsymbol{J}$ and $\boldsymbol{M}$ of equation (14) are

$$\boldsymbol{J}^0 \equiv \lim_{r\to 0} = \boldsymbol{J} \qquad \boldsymbol{M}^0 \equiv \lim_{r\to 0} = -\mathrm{i}\boldsymbol{n}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r} + \frac{1}{r}\boldsymbol{n}\times\boldsymbol{L}. \tag{52}$$

Here the facts that $\boldsymbol{J}\times\boldsymbol{n} + \boldsymbol{n}\times\boldsymbol{J} = 2\mathrm{i}\boldsymbol{n}$ ($\boldsymbol{n}$ is a vector operator with respect to $\boldsymbol{J}$), and that $\boldsymbol{L} = \boldsymbol{J} - \boldsymbol{S}$ have been used. Recalling that $\boldsymbol{p} = -\mathrm{i}\boldsymbol{n}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\boldsymbol{n}\times\boldsymbol{L}$, one can then find that

$$\boldsymbol{J}^0 \equiv \lim_{r\to 0} = \boldsymbol{J} \qquad \boldsymbol{M}^0 \equiv \lim_{r\to 0} = r\boldsymbol{p}\frac{1}{r}. \tag{53}$$

Now the operators $\boldsymbol{p}$ and $\boldsymbol{J}$ satisfy the commutation relations of the Lie algebra of the group $E(3)$, the Euclidean group in three dimensions, and so

$$[J_j, J_k] = \mathrm{i}\epsilon_{jkl}J_l \qquad [J_j, p_k] = \mathrm{i}\epsilon_{jkl}p_l \qquad [p_j, p_k] = 0. \tag{54}$$

Hence

$$[J^0_j, J^0_k] = \mathrm{i}\epsilon_{jkl}J^0_l \qquad [J^0_j, M^0_k] = \mathrm{i}\epsilon_{jkl}M^0_l \qquad [M^0_j, M^0_k] = 0. \tag{55}$$

Thus when performing the limit the $e(3)$ algebra results as a contraction of the $SO(3,1)$ algebra. This result is evident as for $r \to 0$ we obtain the point with coordinates $(x^0, x^1, x^2, x^3) = (1, 0, 0, 0)$ on the double-sheeted hyperboloid. Hence in this case instead of a parametrization of the upper sheet of the hyperboloid, we have a parametrization of its tangent plane at the point $(1, 0, 0, 0)$, which is isomorphic to $\boldsymbol{R}^3$, the three-dimensional Euclidean space.

It is well known that the Casimir operators of $e(3)$ are $\boldsymbol{p}^2$, and $\boldsymbol{J}\boldsymbol{p}$. Since $\boldsymbol{p}\boldsymbol{L} = 0$, we expect that the short-distance form of our Casimir operators then can be expressed in terms of the quantities $\boldsymbol{p}^2$ and $\boldsymbol{S}\boldsymbol{p}$. This is indeed the case as can be seen also from the short-distance limit of equations (21) and (22). Taking the limits $r \to 0$ we get

$$\mathcal{C}^0_1 \equiv \lim_{r\to 0}\mathcal{C}_1 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} - 1 + (\boldsymbol{S}\boldsymbol{n})^2 - \frac{\mathcal{A} + \mathcal{B} - 2\Sigma_1}{r^2} \tag{56}$$

and

$$\mathcal{C}_2^0 \equiv \lim_{r\to 0} \mathcal{C}_2 = -\mathrm{i}\boldsymbol{S}\boldsymbol{n}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\Sigma_2}{r}. \qquad (57)$$

By virtue of equations (16) and (17), $\mathcal{A}+\mathcal{B}-2\Sigma_1 = \boldsymbol{L}^2$, and $\Sigma_2 = \boldsymbol{S}(\boldsymbol{n}\times\boldsymbol{L}) - \mathrm{i}\boldsymbol{S}\boldsymbol{n}$, and therefrom

$$\mathcal{C}_1^0 \equiv -r\boldsymbol{p}^2\frac{1}{r} - 1 + (\boldsymbol{S}\boldsymbol{n})^2 \qquad \mathcal{C}_2^0 \equiv r\boldsymbol{S}\boldsymbol{p}\frac{1}{r}. \qquad (58)$$

Hence as we expected the $e(3)$ Casimir operators appear in the short-distance limit as it has to be.

Now let us consider the eigenvalue problems of our Casimir operators. The operators $\boldsymbol{S}\boldsymbol{n}$ and $\boldsymbol{S}\boldsymbol{p}$ have the eigenvalues λ and λk respectively. From equation (48) and repeating the reasoning in the following paragraph we can again conclude that equation (49) holds. Hence the eigenvalue problem of $\mathcal{C}_1$ yields

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{\boldsymbol{L}^2}{r^2}\right)\psi^0_{k\nu;ls\lambda}(r) = 0 \qquad (59)$$

which is just the usual differential equation of Bessel functions for all values of $\lambda = \pm\nu$ describing the free dynamics in $\boldsymbol{R}^3$.

5. The coupled channel problem

To find the solutions of the eigenvalue problems involving the original Casimir operators, equations (21) and (22), first we need the matrix elements of the operators $\mathcal{A}+\mathcal{B}$, Σ_1 and Σ_2 in the basis given by the functions $\mathcal{D}^{js}_{m\lambda}(\theta,\varphi)$ of equations (36) and (39). Consider the operator $\Sigma_2 = (\boldsymbol{S}\times\boldsymbol{n})\boldsymbol{J}$. Using the transformation $U(\theta,\varphi)$ we get

$$\Sigma_2' = U^\dagger(\theta,\varphi)\boldsymbol{S}\times\boldsymbol{n}U(\theta,\varphi)\boldsymbol{J}' \qquad (60)$$

where $\boldsymbol{J}'$ is given by equation (34). So to find the explicit form of the operator Σ_2' we have to calculate $U^\dagger(\theta,\varphi)\boldsymbol{S}\times\boldsymbol{n}U(\theta,\varphi)$. It is straightforward to show that

$$U^\dagger(\theta,\varphi)\boldsymbol{S}\times\boldsymbol{n}U(\theta,\varphi) = \boldsymbol{E}_1 S_1 + \boldsymbol{E}_2 S_2 \qquad (61)$$

where the vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ can be expressed in terms of the components of the unit vector $(n_1,n_2,n_3) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ by

$$\boldsymbol{E}_1 = \begin{pmatrix} \frac{n_1 n_2}{1+n_3} \\ -1+\frac{n_2^2}{1+n_3} \\ n_2 \end{pmatrix} \qquad \boldsymbol{E}_2 = \begin{pmatrix} 1-\frac{n_1^2}{1+n_3} \\ -\frac{n_1 n_2}{1+n_3} \\ -n_1 \end{pmatrix}. \qquad (62)$$

Then by using equation (34) for $\boldsymbol{J}'$, for $\Sigma_2' = (S_1\boldsymbol{E}_1 + S_2\boldsymbol{E}_2)\boldsymbol{J}'$ we obtain

$$\Sigma_2' = U^\dagger\Sigma_2 U = S_2\left(L_1 - \frac{n_1}{1+n_3}J_3\right) - S_1\left(L_2 - \frac{n_2}{1+n_3}J_3\right). \qquad (63)$$

(Notice that $J_3' = J_3 = L_3 + S_3$.) Moreover, since we have $[\Sigma_2', S_3] = \mathrm{i}\Sigma_1'$ one finds

$$\Sigma_1' = U^\dagger\Sigma_1 U = S_1\left(L_1 - \frac{n_1}{1+n_3}\right) + S_2\left(L_2 - \frac{n_2}{1+n_3}\right). \qquad (64)$$

Now consider the action of these operators on the functions

$$\Omega^{js}_{m\lambda}(\theta,\varphi) \equiv U^\dagger(\theta,\varphi)\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) = D^j_{\lambda m}(\varphi,-\theta,-\varphi)\chi^s_\lambda. \qquad (65)$$

Introducing the complex linear combinations, $\Sigma'_\pm = \Sigma'_1 \pm \mathrm{i}\Sigma'_2$, and rewriting them in terms of θ and φ, we have to calculate the action of operators of the form

$$\Sigma'_\pm \Omega^{js}_{m\lambda}(\theta,\varphi) = \mp S_\pm \mathrm{e}^{\mp\mathrm{i}\varphi}\left(\frac{\partial}{\partial\theta} \mp \frac{\cos\theta S_3 - J_3}{\sin\theta}\right)\Omega^{js}_{m\lambda}(\theta,\varphi). \tag{66}$$

Since $J_3 = -\mathrm{i}\partial_\varphi + S_3$ and the dependence of Wigner's function $D^j_{\lambda m}$ on φ is given by the factor $\mathrm{e}^{\mathrm{i}(m-\lambda)\varphi}$ we find $J_3 D^j_{\lambda m} = m D^j_{\lambda m}$. Recall that $S_3\chi^s_\lambda = \lambda\chi^s_\lambda$, and note that we have used the Varshalovich convention [10] for Wigner's D function. The result is

$$\Sigma'_\pm \Omega^{js}_{m\lambda}(\theta,\varphi) = \mp \mathrm{e}^{\mp\mathrm{i}\varphi}\left(\frac{\partial}{\partial\theta} \mp \frac{\lambda\cos\theta - m}{\sin\theta}\right) D^j_{\lambda m}(\varphi,-\theta,-\varphi)(S_\pm\chi^s_\lambda). \tag{67}$$

Furthermore identities given in [10] (see equations (4), (5) p 94) enable us to deduce that

$$\left(\partial_\beta \mp \frac{M' - M\cos\beta}{\sin\beta}\right) D^J_{MM'}(\alpha,\beta,\gamma) = \mp\sqrt{(J\pm M)(J\mp M+1)}\mathrm{e}^{\mp\mathrm{i}\alpha} D^J_{M\mp 1 M'}(\alpha,\beta,\gamma). \tag{68}$$

Then with $\alpha = \varphi$, $\beta = -\theta$, $J = j$, $M = \lambda$, $M' = m$ and using

$$S_\pm\chi^s_\lambda = \sqrt{(s\mp\lambda)(s\pm\lambda+1)}\chi^s_{\lambda\pm1} \tag{69}$$

we finally get

$$\Sigma'_\pm \Omega^{js}_{m\lambda}(\theta,\varphi) = \sqrt{(s\mp\lambda)(s\pm\lambda+1)(j\mp\lambda)(j\pm\lambda+1)}\Omega^{js}_{m\lambda\pm1}(\theta,\varphi). \tag{70}$$

The action of the operators $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{D}^{js}_{m\lambda}$ can be calculated easily given that $[S^2, U] = 0$, from such the functions $\Omega^{js}_{m\lambda}$ become simply the eigenvectors of $\mathcal{A}'$ and $\mathcal{B}'$ with eigenvalues $j(j+1) - \lambda^2$ and $s(s+1) - \lambda^2$.

Finally the actions of the operators $\mathcal{A}$, $\mathcal{B}$, $\Sigma_\pm$ on the functions $\mathcal{D}^{js}_{m\lambda}(\theta,\varphi)$ are

$$\Sigma_\pm \mathcal{D}^{js}_{m\lambda}(\theta,\varphi) = \sqrt{(s\mp\lambda)(s\pm\lambda+1)(j\mp\lambda)(j\pm\lambda+1)}\mathcal{D}^{js}_{m\lambda\pm1}(\theta,\varphi) \tag{71}$$

$$\mathcal{A}\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) = (j(j+1) - \lambda^2)\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) \tag{72}$$

$$\mathcal{B}\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) = (s(s+1) - \lambda^2)\mathcal{D}^{js}_{m\lambda}(\theta,\varphi). \tag{73}$$

Hence in this basis the operators $\Sigma_\pm$ couple states of different helicity.

Now we recast the wavefunction in the form

$$\Psi_{k\nu;jms\lambda}(r,\theta,\varphi) \equiv \psi^k_{js;\lambda\nu}(r)\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) \tag{74}$$

where the labelling follows the definitions given with equation (31). Moreover, the channel wavefunctions $\psi^k_{js;\lambda\nu}(r)$ for the fixed values of k, j and s are square matrices labelled by the pair of indices $\lambda\nu$.

There are some properties of $\psi^k_{js;\lambda\nu}(r)$ to be defined. The first is the dimension of the square matrix $\psi_{\lambda\nu}(r)$. As we will show the dimension is $2s+1$ when $s \leqslant j$, and $2j+1$ when $j < s$. The second property to be proved is that this matrix is a symmetrical one, as was assumed in the discussion of the asymptotic behaviour of our wavefunction. To clarify such issues we have to understand the group theoretical meaning of $\psi^k_{js;\lambda\nu}(r)$.

To do so first we write down the eigenequations for $\psi_{\lambda\nu}(r)$. Using the explicit form equations (21) and (22) in the basis as given by $\mathcal{D}^{js}_{m\lambda}$ we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 + a\,dS_3^2 - \frac{Z - 2X\cosh r}{\sinh^2 r}\right)_{\lambda\lambda'}\psi_{\lambda'\nu}(r) = 0 \tag{75}$$

$$\left(-\mathrm{i}S_3\frac{\mathrm{d}}{\mathrm{d}r} - \frac{Y}{\sinh r}\right)_{\lambda\lambda'}\psi_{\lambda'\nu}(r) = k\psi_{\lambda\lambda'}(r)(S_3)_{\lambda'\nu}. \tag{76}$$

Here

$$(Z\psi)_{\lambda\nu} = (j(j+1) + s(s+1) - 2\lambda^2)\psi_{\lambda\nu} \tag{77}$$

$$((X \pm \mathrm{i}Y)\psi)_{\lambda\nu} = \sqrt{(s \mp \lambda)(s \pm \lambda + 1)(j \mp \lambda)(j \pm \lambda + 1)}\,\psi_{\lambda\pm1\nu} \tag{78}$$

$$a\,dS_3^2\psi \equiv [S_3, \psi] = (\lambda^2 - \nu^2)\psi. \tag{79}$$

Equation (75) is a 'Schrödinger-like' equation, describing a coupled channel scattering problem with a $(2s+1)\times(2s+1)$ matrix-interaction term of $-\frac{Z-2X\cosh r}{\sinh^2 r}$. According to equations (77) and (78), this interaction term has a tridiagonal matrix form. Notice that this equation (apart from the spin $\frac{1}{2}$ case) is *not* a Schrödinger equation describing a *single* coupled channel scattering problem, but rather an equation describing a *collection* of coupled channel scattering problems. This can be seen from the presence of the term $adS_3^2\psi$ of equation (79) in (75), although it vanishes when $\lambda = \pm\nu$, or $\lambda = \nu = 0$. In section 2 we found that this restriction physically means that we deal with a helicity scattering process. In light of this we can say that equation (75) describes a *collection* of helicity scattering problems. This also means that we should be able to reduce the tridiagonal form of the interaction term to a block-diagonal form containing the 2×2 helicity blocks for $\lambda = \pm\nu$, and the 1×1 block for $\lambda = \nu = 0$. We postpone the discussion of this problem to a later section.

Now we turn back to a group theoretical description of the matrix ψ. First notice that equation (75) has a form similar to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} - \left(J + \frac{1}{2}\right)^2 + \frac{M^2 + M'^2 - \frac{1}{4} - 2MM'\cos\theta}{\sin^2\theta}\right)\sqrt{\sin\theta}\, d^J_{MM'}(\theta) = 0 \tag{80}$$

which is associated with the problem of the Casimir operator of the $SO(3)$ algebra. Indeed this equation derives from

$$J^2 D^J_{MM'}(\alpha,\beta,\gamma) = J(J+1)D^J_{MM'}(\alpha,\beta,\gamma) \qquad D^J_{MM'} = \mathrm{e}^{-\mathrm{i}M\alpha} d^J_{MM'}(\beta)\mathrm{e}^{-\mathrm{i}M'\gamma} \tag{81}$$

on using the explicit form [10] of $\boldsymbol{J}^2$ expressed in terms of (α,β,γ) with $\beta = \theta$, and then employing a similarity transformation with $\sqrt{\sin\theta}$.

The essential difference between equations (80) and (75) is that matrices appear in equation (75) instead of integers and half integers. Otherwise equation (75) seems to be the hyperbolic analogue of equation (80). Moreover, both of these equations arise from the eigenvalue problem of a *quadratic Casimir operator*. Equation (80) arises from the Casimir of $SO(3)$, while equation (75) arises from the Casimir of $SO(3,1)$. However, $SO(3,1)$ is a noncompact algebra of rank two. Accordingly we have the possibility to describe also scattering states by using a series of irreducible representations indexed by *continuously* changing labels, and, as it is of rank two, we have an additional number (in our case it is ν) to label the possible scattering channels besides k. Based on these observations, we expect $\psi^k_{js;\lambda\nu}(r)$ to be a matrix-valued generalization of Wigner's $d^J_{MM'}(\theta)$ function. Here the notation is very instructive. The upper index k (together with the lower index ν) of ψ labels the irrep of $SO(3,1)$, likewise the upper index J of d labels the irrep of $SO(3)$. The lower indices js label the $SO(3)$ basis in which the matrix elements are calculated. Likewise the lower indices MM' label the corresponding $SO(2)$ basis vectors. However, in the $SO(3,1)$ case we have another subalgebra—namely $SO(2)\subset SO(3)\subset SO(3,1)$. This accounts for the label λ tagging the particular basis vectors within the irrep labelled by s. Note that we use the other $SO(3,1)$ label ν as a *subscript* accompanying the $SO(2)$ index λ. This is a convenient notation as the matrix character of $\psi_{\lambda\nu}$ has to be reflected in the special behaviour of the $SO(3,1)$ label ν and the $SO(2)$ label λ under exchange. Then as

$$d^J_{MM'}(\theta) = \langle JM|\mathrm{e}^{-\mathrm{i}\theta J_2}|JM'\rangle \tag{82}$$

we seek to represent $\psi^k_{js;\lambda\nu}(r)$ similarly as

$$\psi^k_{js;\lambda\nu}(r) = \langle k\nu; j\lambda|\mathrm{e}^{-\mathrm{i}rM_3}|k\nu; s\lambda\rangle. \tag{83}$$

We have chosen the generator M_3 since it commutes with J_3 by virtue of equation (5). Moreover, notice that instead of the pair (j, m) the pair (j, λ) is used in equation (83). Since $J_3' = J_3 = L_3 + S_3$, the eigenvalue m is related to λ by $m = l + \lambda$. Moreover, we know [7] that

$$(k, \nu) = \bigoplus_{j=|\nu|}^{\infty} (j, m) \tag{84}$$

meaning that in the restriction of the $SO(3, 1)$ representation (k, ν) to $SO(3)$, only those representations occur for which $j = |\nu|, |\nu| + 1, \ldots.$ While the representation space is infinite dimensional, it is built from the finite-dimensional representation spaces of $SO(3)$. According to equation (40) we have the additional restriction $j \geqslant |\lambda|$. Hence in order that these restrictions be consistent, we assume $s \leqslant j$. In that case since $|\lambda| \leqslant s$ and $|\nu| \leqslant s$, both of our restrictions are satisfied. Moreover, both of the $SO(3)$ representations labelled by j and s occur (with multiplicity one) in the irrep (k, ν) of $SO(3, 1)$. Hence the matrix element given by equation (83) makes sense.

It is important to recall how M_3 acts on the basis $|j\lambda\rangle$ (a similar formula holds for $|s\lambda\rangle$ as well.) The result [7] is

$$M_3|j\lambda\rangle = \sqrt{j^2 - \lambda^2}c_j|j - 1\lambda\rangle + \lambda a_j|j\lambda\rangle - \sqrt{(j + 1)^2 - \lambda^2}c_{j+1}|j + 1\lambda\rangle \tag{85}$$

where

$$c_j = \frac{\mathrm{i}}{j}\sqrt{\frac{(j^2 - \nu^2)(j^2 + k^2)}{4j^2 - 1}} \qquad a_j = \frac{k\nu}{j(j + 1)}. \tag{86}$$

In this equation interchange of the labels ν and λ does not change the action of M_3. Thus for $j \geqslant s$ our wave-matrix $\psi^k_{js;\lambda\nu}(r)$ in the pair of indices $\lambda\nu$ is a $(2s + 1) \times (2s + 1)$ *symmetric* matrix. If however, $j \leqslant s$, then $-j \leqslant \lambda, \nu \leqslant j$, and the wave-matrix then is $(2j+1) \times (2j+1)$ symmetric.

The foregoing considerations were built on the formal similarity between our channel wavefunction $\psi^k_{js;\lambda\nu}(r)$ and Wigner's $d^J_{MM'}$ function. But we also must ensure that our wavefunction (defined by equation (83)) satisfies equations (75) and (76). Since $d^j_{MM'}(\theta)$ can be expressed in terms of the Jacobi polynomials [10], a guess is that $\psi^k_{js;\lambda\nu}(r)$ can be expressed in terms of a matrix-valued generalization of the (hyperbolic) Jacobi polynomials. In that search the detailed review of [11] of matrix-valued special functions for the groups $SO(n, 1)$ is very useful. For example we have found that for $n = 3$ equation (75) is related to the defining equation for the hyperbolic generalization of the matrix-Jacobi polynomials.

To be more specific consider equation (75). With a change of variable ($\rho = \cosh r$) and a similarity transformation by $\sinh r$ this can be written

$$\left((\rho^2 - 1)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + 3\rho\frac{\mathrm{d}}{\mathrm{d}\rho} + k^2 - adS_3^2 - \frac{Z - 2\rho X}{\rho^2 - 1}\right)\mathcal{P}(\rho) = 0 \tag{87}$$

where $\mathcal{P}(\rho) \equiv \mathcal{P}(\cosh r)$ is related to the wavefunction defined by equation (83) (for simplicity here the indices have been omitted), by

$$\psi(r) = \sinh r\mathcal{P}(\cosh r). \tag{88}$$

This equation is precisely the hyperbolic analogue of equation (3) on p 426 of [11]. The function $\mathcal{P}$ is the hyperbolic equivalent of the 'Jacobi function with matrix indices' as introduced in [11]. Hence our coupled channel wavefunction, given in equation (83), really satisfies equation (75). Then since the corresponding differential operators (related to $\mathcal{C}_1$ and $\mathcal{C}_2$) commute it satisfies equation (76) too.

6. The coupled channel wavefunction

In the previous section the group theoretical meaning of our coupled channel wavefunction solving equations (75) and (76) was established. Next we seek how to express the abstract matrix wavefunction equation (83) in terms of conventional special functions. A scheme of solving the matrix differential equation (87) was presented in [11]. That formulation used notions of the theory of functions of ordered operators. Here we merely present the solution of equation (87). Proofs are given elsewhere [11].

First fix the relationship between j and s to be $j \geqslant s$. The $j \leqslant s$ case is a straightforward rerun of the method outlined below. Next define new $(2s+1) \times (2s+1)$ matrices R and Q by fixing their action as

$$R_{\pm}\Phi_{\lambda\nu} = \sqrt{\frac{(j \pm \lambda + 1)}{(j \mp \lambda)}(s \mp \lambda)(s \pm \lambda + 1)}\,\Phi_{\lambda\pm 1\nu} \qquad R = \tfrac{1}{2}(R_+ + R_-) \tag{89}$$

$$Q_{\mp}\Phi_{\lambda\nu} = \frac{\mathrm{i}k \mp \lambda - 1}{\mathrm{i}k \pm \lambda}\sqrt{\frac{(j \pm \lambda + 1)}{(j \mp \lambda)}(s \mp \lambda)(s \pm \lambda + 1)}\,\Phi_{\lambda\pm 1\nu} \qquad Q = \tfrac{1}{2}(Q_+ + Q_-). \tag{90}$$

Notice that these matrices are related to X as defined in equation (78). We will need the spectral projectors R_a and Q_b $a, b = -s, \ldots, s$ of these matrices defined by

$$R = \sum_{a=-s}^{s} r_a R_a \qquad Q = \sum_{b=-s}^{s} q_b Q_b. \tag{91}$$

Here r_a and q_b are the corresponding eigenvalues of the matrices R and Q. According to theorem 1 on p 438 of [11], these nondegenerate eigenvalues are

$$r_a = a \qquad q_b = b \qquad a, b = -s, \ldots, s \tag{92}$$

which lie in the usual range $-s \leqslant r_a, q_b \leqslant s$. With them the spectral projectors can be expressed as

$$R_a = \prod_{c=-s, c\neq a}^{s} \frac{R - r_c E}{r_a - r_c} \qquad Q_b = \prod_{c=-s, c\neq b}^{s} \frac{Q - q_c E}{q_b - q_c}. \tag{93}$$

The solution of equation (87) then follows. According to theorem 4 on p 430 of [11], that solution can be expressed in the form

$$\mathcal{P}(\rho) = \sum_{a,b=-s}^{s} R_a Q_b C f_{r_a,q_b}(\rho) \tag{94}$$

where the functions $f^{kj}_{r_a,q_b}(\rho)$ satisfy the differential equation

$$\left((\rho^2 - 1)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + (3\rho + r - q)\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{j(j+1) + rq - \rho(jr + (j+1)q)}{\rho^2 - 1} + k^2\right) f^{kj}_{r,q}(\rho) = 0 \tag{95}$$

(for simplicity the subscripts of r and q are left implicit). As was shown [11] this equation can be reduced to that of the ordinary Jacobi polynomials, so that

$$f^{kj}_{r_a q_b}(r) = \left(\sinh\frac{r}{2}\right)^{j-q_b}\left(\cosh\frac{r}{2}\right)^{j+q_b} P^{\alpha_{ab},\beta_{ab}}_{ik-j-1}(\cosh r) \tag{96}$$

where

$$\alpha_{ab} = j + \tfrac{1}{2} - \tfrac{1}{2}(r_a + q_b) \qquad \beta_{ab} = j + \tfrac{1}{2} + \tfrac{1}{2}(r_a - q_b) \tag{97}$$

and $\rho = \cosh r$. C is a $(2s+1) \times (2s+1)$ numerical matrix [11], that we fix case by case to ensure the symmetric character of ψ.

It is more convenient to express these functions in terms of the functions $\mathcal{B}^l_{mn}(\cosh r)$, the $SO(2,1)$ analogues of the usual $SO(3)$ Wigner functions. The definitions and properties of these functions are detailed in [12]. Of import here is that they satisfy the differential equation [12]

$$\left((\rho^2-1)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + 2\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{m^2+n^2-2mn\rho}{\rho^2-1} - l(l+1)\right)\mathcal{B}^l_{mn}(\rho) = 0. \qquad (98)$$

Comparing with equation (95) reveals that

$$\mathcal{B}^l_{mn}(\cosh r) = \left(\sinh\frac{r}{2}\right)^{m-n}\left(\cosh\frac{r}{2}\right)^{m+n} P^{(m-n,m+n)}_{l-m}(\cosh r) \qquad (99)$$

or

$$f^{kj}_{r_a q_b}(r) = \left(\sinh\frac{r}{2}\right)^{\frac{1}{2}(r_a-q_b-1)}\left(\cosh\frac{r}{2}\right)^{\frac{1}{2}(q_b-r_a-1)}\mathcal{B}^{-\frac{1}{2}+ik}_{j+\frac{1}{2},\frac{1}{2}(r_a+q_b)}(\cosh r). \qquad (100)$$

Thus using the relation equation (88), our wavefunction can be written in the form

$$\psi^k_{js} = \sum_{a,b=-s}^{s} R_a Q_b C \Phi^{kj}_{r_a q_b}(r) \qquad -s \leqslant r_a, q_b \leqslant s \qquad (101)$$

where

$$\Phi^{kj}_{r_a q_b}(r) = \sqrt{\sinh r}\left(\tanh\frac{r}{2}\right)^{\frac{1}{2}(r_a-q_b)}\mathcal{B}^{-\frac{1}{2}+ik}_{j+\frac{1}{2},\frac{1}{2}(r_a+q_b)}(\cosh r). \qquad (102)$$

For convenience in equation (101) the matrix indices of $\psi^k_{js;\lambda\nu}(r)$ carried by the matrices $(R_a)_{\lambda\nu}$ and $(Q_b)_{\lambda\nu}$ were left implicit.

To complete our solution as given by equations (101) and (102) $\mathcal{B}^l_{mn}(\cosh r)$ can be expressed in terms of the hypergeometric function [12].

$$\mathcal{B}^l_{mn} = \frac{\Gamma(l-n+1)}{\Gamma(l-m+1)(m-n)!}\left(\cosh\frac{r}{2}\right)^{m+n} \times \left(\sinh\frac{r}{2}\right)^{m-n} F\left(l+m+1, m-l; m-n+1; -\sinh^2\frac{r}{2}\right) \qquad (103)$$

where it is understood that

$$l = -\tfrac{1}{2} + \mathrm{i}k \qquad m = j + \tfrac{1}{2} \qquad n \equiv n_{ab} = \tfrac{1}{2}(r_a + q_b). \qquad (104)$$

Note that equation (103) is only valid for $m \geqslant n$ [12]. However, according to our restriction $j \geqslant s$ and the definitions given in equation (104) this condition is satisfied.

Finally, note that our total matrix-valued coupled channel wavefunction, equation (101), is regular at the origin, as it must be. Thus to have the explicit solution of the coupled channel wavefunction, the only task left is to calculate the explicit form of the spectral projectors R_a and Q_b. We do so in the next three subsections for the particular choices of $s = \frac{1}{2}$, 1 and $\frac{3}{2}$.

6.1. The $s = \frac{1}{2}$ case

This very special case has already been investigated previously [5]. However, it is also the simplest case at hand, and so it is very instructive to consider this first as the clearest example of how our construction works. The interaction term of the coupled channel problem can be calculated by using the form of the matrices Z and X as specified in equations (77) and (78) respectively. The final result for the potential to be used then in the coupled channel Schrödinger equation is

$$V_{\lambda\nu}(r) = \begin{pmatrix} \frac{(j+\frac{1}{2})^2}{\sinh^2 r} & -\frac{(j+\frac{1}{2})\cosh r}{\sinh^2 r} \\ -\frac{(j+\frac{1}{2})\cosh r}{\sinh^2 r} & \frac{(j+\frac{1}{2})^2}{\sinh^2 r} \end{pmatrix}. \tag{105}$$

Then as $adS_3^2\psi = 0$ the wavefunction sought is a solution of a Schrödinger equation with just the matrix-valued interaction term of equation (105).

To proceed as $s = \frac{1}{2}$, the labels a, b can have the values $\pm\frac{1}{2}$, and we have to calculate the explicit form of the 2×2 matrices of R and Q. Using the definitions in equations (89) and (90), these matrices are identical as are their eigenvalues, i.e.

$$R = Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad r_{\pm\frac{1}{2}} = \pm\tfrac{1}{2} \quad q_{\pm\frac{1}{2}} = \pm\tfrac{1}{2}. \tag{106}$$

The spectral projectors of equation (93) then follow and are

$$R_{\frac{1}{2}} = Q_{\frac{1}{2}} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad R_{-\frac{1}{2}} = Q_{-\frac{1}{2}} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{107}$$

The projector properties are satisfied and so

$$R_{\frac{1}{2}}Q_{\frac{1}{2}} = R_{\frac{1}{2}} \qquad R_{-\frac{1}{2}}S_{-\frac{1}{2}} = R_{-\frac{1}{2}} \qquad R_{\frac{1}{2}}Q_{-\frac{1}{2}} = R_{-\frac{1}{2}}Q_{\frac{1}{2}} = 0. \tag{108}$$

The total wavefunction as given by equation (101) then is

$$\psi^k_{j\frac{1}{2}}(r) = \sum_{a,b=\pm\frac{1}{2}} R_a Q_b \Phi^{kj}_{r_a q_b}(r) = R_{\frac{1}{2}}\Phi^{kj}_{\frac{1}{2}\frac{1}{2}}(r) + R_{-\frac{1}{2}}\Phi^{kj}_{-\frac{1}{2}-\frac{1}{2}}(r). \tag{109}$$

Notice that in this case the restriction $j \geqslant s$ covers *all cases* since $|l - s| \leqslant j \leqslant l + s$, so for $s = \frac{1}{2}$, $j \geqslant \frac{1}{2}$ in all cases. Finally from equations (101) and (102), the explicit form of our coupled channel wavefunction is

$$\psi^k_{j\frac{1}{2};\lambda\nu}(r) = \tfrac{1}{2}\sqrt{\sinh r}\begin{pmatrix} \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},\frac{1}{2}}(r) + \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},-\frac{1}{2}}(r) & \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},\frac{1}{2}}(r) - \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},-\frac{1}{2}}(r) \\ \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},\frac{1}{2}}(r) - \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},-\frac{1}{2}}(r) & \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},\frac{1}{2}}(r) + \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},-\frac{1}{2}}(r) \end{pmatrix}. \tag{110}$$

Notice that as ψ is a symmetric matrix, the numerical matrix C in equation (101) can be chosen to be the 2×2 unit matrix. Also the columns of the symmetric matrix ψ transform according to different irreducible representations of $SO(3,1)$. The first column of equation (110) transforms with respect to the irrep $(\mathrm{i}k, \frac{1}{2})$ of $SO(3,1)$ and the second according to $(\mathrm{i}k, -\frac{1}{2})$. These representations are mirror conjugated to each other.

6.2. *The $s = 1$ case*

For spin 1 particle scattering the interaction term of the coupled channel problem again can be calculated by using the form of the matrices Z and X given in equations (77) and (78). The result to be used in equation (75), the coupled channel Schrödinger equation, is

$$V_{\lambda\nu}(r) = \begin{pmatrix} \frac{j(j+1)}{\sinh^2 r} & \frac{-\sqrt{j(j+1)}\cosh r}{\sinh^2 r} & 0 \\ \frac{-\sqrt{j(j+1)}\cosh r}{\sinh^2 r} & \frac{j(j+1)+2}{\sinh^2 r} & \frac{-\sqrt{j(j+1)}\cosh r}{\sinh^2 r} \\ 0 & \frac{-\sqrt{j(j+1)}\cosh r}{\sinh^2 r} & \frac{j(j+1)}{\sinh^2 r} \end{pmatrix}. \tag{111}$$

For the spin 1 case $a, b = -1, 0, 1$, and it is useful to introduce the variables

$$\alpha = \sqrt{\frac{j}{j+1}} \qquad \beta = \frac{\mathrm{i}k}{\mathrm{i}k - 1}. \tag{112}$$

Then the matrices R and Q have the form

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \alpha & 0 \\ \alpha^{-1} & 0 & \alpha^{-1} \\ 0 & \alpha & 0 \end{pmatrix} \qquad Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \alpha\beta & 0 \\ (\alpha\beta)^{-1} & 0 & (\alpha\beta)^{-1} \\ 0 & \alpha\beta & 0 \end{pmatrix} \quad (113)$$

which satisfy the property $R^3 = R$ and $Q^3 = Q$. The associated eigenvalues are

$$r_1 = q_1 = 1 \qquad r_0 = q_0 = 0 \qquad r_{-1} = q_{-1} = -1 \quad (114)$$

and the spectral projectors are

$$R_1 = \tfrac{1}{2}R(R+E) \qquad R_0 = (E-R)(E+R) \qquad R_{-1} = \tfrac{1}{2}R(R-E). \quad (115)$$

Identical equations hold for Q_a with R simply replaced by Q. The explicit form of the projectors is

$$R_{\pm 1} = \tfrac{1}{4}\begin{pmatrix} 1 & \pm\sqrt{2}\alpha & 1 \\ \pm\sqrt{2}(\alpha)^{-1} & 2 & \pm\sqrt{2}(\alpha)^{-1} \\ 1 & \pm\sqrt{2}\alpha & 1 \end{pmatrix} \qquad R_0 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \quad (116)$$

In this case to define Q_a replace α by $\alpha\beta$. Since $R_0 Q_{-1} = R_{-1} Q_0 = R_0 Q_1 = R_1 Q_0 = 0$, and $R_0 = Q_0$, we have $R_0 Q_0 = R_0$ and the wavefunction is

$$\psi^k_{j1}(r) = \sqrt{\sinh r}\Bigg(R_1 Q_1 \mathcal{B}_1 + R_{-1} Q_{-1} \mathcal{B}_{-1} + \left(R_0 + R_1 Q_{-1} \tanh\frac{r}{2} + R_{-1} Q_1 \coth\frac{r}{2} \right)\mathcal{B}_0 \Bigg) \quad (117)$$

where for brevity

$$\mathcal{B}_\mu \equiv \mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{j+\frac{1}{2},\mu}(\cosh r) \qquad \mu = -1, 0, +1. \quad (118)$$

With the matrix C specified by

$$C \equiv \begin{pmatrix} \mathrm{i}k & 0 & 0 \\ 0 & \mathrm{i}k - 1 & 0 \\ 0 & 0 & \mathrm{i}k \end{pmatrix} \quad (119)$$

the matrix $\psi^k_{j1;\lambda\nu}(r)$ is symmetric with components

$$\psi^k_{j1;1\pm 1}(r) = \psi^k_{j1;-1\mp 1}(r) = \frac{\sqrt{\sinh r}}{4}\left[\left(\mathrm{i}k - \frac{1}{2} \right)(\mathcal{B}_1(r) + \mathcal{B}_{-1}(r)) + (\coth r \pm 2\mathrm{i}k)\mathcal{B}_0(r) \right] \quad (120)$$

$$\psi^k_{j1;00}(r) = \frac{\sqrt{\sinh r}}{2}\left[\left(\mathrm{i}k - \frac{1}{2} \right)(\mathcal{B}_1(r) + \mathcal{B}_{-1}(r)) - \coth r\,\mathcal{B}_0(r) \right] \quad (121)$$

$$\psi^k_{j1,10}(r) = \psi^k_{j1,01}(r) = \psi^k_{j1,-10}(r) = \psi^k_{j1,0-1}(r) = \sqrt{\frac{j(j+1)}{2\sinh r}}\,\mathcal{B}_0(r). \quad (122)$$

As a check of our method, in the appendix we show that these wavefunctions satisfy equation (76), the eigenvalue problem of the other Casimir operator ($\mathcal{C}_2$).

6.3. The $s = \frac{3}{2}$ case

In this case the interaction term $V_{\lambda\nu}(r)$ is a 4×4 matrix of the form

$$\begin{pmatrix} \frac{j(j+1)+\frac{3}{2}}{\sinh^2 r} & -\frac{\sqrt{3(j+\frac{3}{2})(j-\frac{1}{2})}\cosh r}{\sinh^2 r} & 0 & 0 \\ -\frac{\sqrt{3(j+\frac{3}{2})(j-\frac{1}{2})}\cosh r}{\sinh^2 r} & \frac{j(j+1)+\frac{7}{2}}{\sinh^2 r} & -\frac{(2j+1)\cosh r}{\sinh^2 r} & 0 \\ 0 & -\frac{(2j+1)\cosh r}{\sinh^2 r} & \frac{j(j+1)+\frac{7}{2}}{\sinh^2 r} & -\frac{\sqrt{3(j+\frac{3}{2})(j-\frac{1}{2})}\cosh r}{\sinh^2 r} \\ 0 & 0 & -\frac{\sqrt{3(j+\frac{3}{2})(j-\frac{1}{2})}\cosh r}{\sinh^2 r} & \frac{j(j+1)+\frac{3}{2}}{\sinh^2 r} \end{pmatrix} \tag{123}$$

as results by calculating the matrices Z and X. The $2s + 1 = 4$ eigenvalues for the matrices R and Q lie in the expected range

$$-\tfrac{3}{2} \leqslant r, q \leqslant \tfrac{3}{2}. \tag{124}$$

The corresponding projectors are

$$R_{\pm\frac{3}{2}} = \pm\tfrac{1}{6}(R - \tfrac{1}{2}E)(R + \tfrac{1}{2}E)(R \pm \tfrac{3}{2}E) \tag{125}$$

$$R_{\pm\frac{1}{2}} = \mp\tfrac{1}{2}(R - \tfrac{3}{2}E)(R + \tfrac{3}{2}E)(R \pm \tfrac{1}{2}E) \tag{126}$$

and similarly for Q. Introducing again new variables

$$\alpha = \sqrt{\frac{j + \frac{3}{2}}{j - \frac{1}{2}}} \qquad \beta = \frac{\mathrm{i}k - \frac{3}{2}}{\mathrm{i}k + \frac{1}{2}} \tag{127}$$

the R_a projectors are of the form

$$R_{\pm\frac{3}{2}} = \frac{1}{8}\begin{pmatrix} 1 & \pm\sqrt{3}\alpha^{-1} & \sqrt{3}\alpha^{-1} & \pm 1 \\ \pm\sqrt{3}\alpha & 3 & \pm 3 & \sqrt{3}\alpha \\ \sqrt{3}\alpha & \pm 3 & 3 & \pm\sqrt{3}\alpha \\ \pm 1 & \sqrt{3}\alpha^{-1} & \pm\sqrt{3}\alpha^{-1} & 1 \end{pmatrix} \tag{128}$$

$$R_{\pm\frac{1}{2}} = \frac{1}{8}\begin{pmatrix} 3 & \pm\sqrt{3}\alpha^{-1} & -\sqrt{3}\alpha^{-1} & \mp 3 \\ \pm\sqrt{3}\alpha & 1 & \mp 1 & -\sqrt{3}\alpha \\ -\sqrt{3}\alpha & \mp 1 & 1 & \pm\sqrt{3}\alpha \\ \mp 3 & -\sqrt{3}\alpha^{-1} & \pm\sqrt{3}\alpha^{-1} & 3 \end{pmatrix}. \tag{129}$$

For the Q projectors we have similar expressions with α again replaced by $\alpha\beta$. Then it is straightforward to show that

$$R_{\frac{3}{2}}Q_{\frac{1}{2}} = R_{\frac{3}{2}}Q_{-\frac{3}{2}} = R_{\frac{1}{2}}Q_{-\frac{1}{2}} = R_{-\frac{1}{2}}Q_{-\frac{3}{2}} = 0. \tag{130}$$

The expressions obtained from those in equation (130) upon exchange of indices also vanish. Hence the structure of the wavefunction is

$$\begin{aligned}\psi^k_{j\frac{3}{2}}(r) = \sqrt{\sinh r}\,(&R_{\frac{3}{2}}Q_{\frac{3}{2}}\mathcal{B}_{\frac{3}{2}}(r) + R_{-\frac{3}{2}}Q_{-\frac{3}{2}}\mathcal{B}_{-\frac{3}{2}}(r) \\ &+ \left(R_{\frac{3}{2}}Q_{-\frac{1}{2}}\tanh\frac{r}{2} + R_{\frac{1}{2}}Q_{\frac{1}{2}} + R_{-\frac{1}{2}}Q_{-\frac{3}{2}}\coth\frac{r}{2}\right)\mathcal{B}_{\frac{1}{2}}(r) \\ &+ \left(R_{\frac{1}{2}}Q_{-\frac{3}{2}}\tanh\frac{r}{2} + R_{-\frac{1}{2}}Q_{-\frac{1}{2}} + R_{-\frac{3}{2}}Q_{\frac{1}{2}}\coth\frac{r}{r}\right)\mathcal{B}_{-\frac{1}{2}}(r)).\end{aligned} \tag{131}$$

Defining the matrix C as

$$C = \begin{pmatrix} \mathrm{i}k + \frac{1}{2} & 0 & 0 & 0 \\ 0 & \mathrm{i}k - \frac{3}{2} & 0 & 0 \\ 0 & 0 & \mathrm{i}k - \frac{3}{2} & 0 \\ 0 & 0 & 0 & \mathrm{i}k + \frac{1}{2} \end{pmatrix} \tag{132}$$

a long but straightforward calculation then yields the final form for the wavefunction components (for simplicity we use the shorthand notation $\psi_{\lambda\nu}$ instead of $\psi^{k}_{j\frac{3}{2};\lambda\nu}(r)$; moreover we omit the obvious r argument of the function $\mathcal{B}(r)$)

$$\psi_{\frac{3}{2}\pm\frac{3}{2}}(r) = \psi_{-\frac{3}{2}\mp\frac{3}{2}} = \tfrac{1}{8}\sqrt{\sinh r}\big((\mathrm{i}k-1)(\mathcal{B}_{\frac{3}{2}} \pm \mathcal{B}_{-\frac{3}{2}}) + 3(\coth r \pm \mathrm{i}k)(\mathcal{B}_{\frac{1}{2}} \pm \mathcal{B}_{-\frac{1}{2}})\big) \tag{133}$$

$$\psi_{\frac{1}{2}\pm\frac{1}{2}}(r) = \psi_{-\frac{1}{2}\mp\frac{1}{2}} = \tfrac{1}{8}\sqrt{\sinh r}\big(3(\mathrm{i}k-1)(\mathcal{B}_{\frac{3}{2}} \pm \mathcal{B}_{-\frac{3}{2}}) - (3\coth r \mp \mathrm{i}k)(\mathcal{B}_{\frac{1}{2}} \pm \mathcal{B}_{-\frac{1}{2}})\big) \tag{134}$$

$$\psi_{\pm\frac{3}{2}\pm\frac{1}{2}}(r) = \psi_{\pm\frac{1}{2}\pm\frac{3}{2}}(r) = \frac{1}{4}\sqrt{\frac{3(j-\frac{1}{2})(j+\frac{3}{2})}{\sinh r}}\big(\mathcal{B}_{\frac{1}{2}} + \mathcal{B}_{-\frac{1}{2}}\big) \tag{135}$$

$$\psi_{\pm\frac{3}{2}\mp\frac{1}{2}}(r) = \psi_{\mp\frac{1}{2}\pm\frac{3}{2}}(r) = \frac{1}{4}\sqrt{\frac{3(j-\frac{1}{2})(j+\frac{3}{2})}{\sinh r}}\big(\mathcal{B}_{\frac{1}{2}} - \mathcal{B}_{-\frac{1}{2}}\big). \tag{136}$$

7. The asymptotic behaviour of the coupled channel wavefunction

In this section we study the asymptotic behaviour of our coupled channel wavefunction as given by equations (101) and (102). Since this wavefunction, according to equation (103), can be expressed in terms of the hypergeometric function, of import for us is the asymptotic behaviour of this function [13].

$$\lim_{|z|\to\infty} F(a,b;c;z) \sim \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(a)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}. \tag{137}$$

In our case we have

$$z(r) = -\sinh^2 r \qquad a = j+1+\mathrm{i}k \qquad b = j+1-\mathrm{i}k \qquad c_{ab} = j+\tfrac{3}{2}-\omega_{ab} \tag{138}$$

where

$$\omega_{ab} \equiv \tfrac{1}{2}(r_a+q_b) \qquad -s \leqslant a, b \leqslant s \qquad r_a = a \qquad q_b = b. \tag{139}$$

It is clear that $-s \leqslant \omega \leqslant s$ as well. Using equations (101)–(103), the asymptotic form is governed by the corresponding form of $\Phi^{kj}_{r_a q_b}(r)$, which can be expressed as

$$\lim_{r\to\infty} \Phi^{kj}_{rq}(r) \sim \frac{\Gamma(\frac{1}{2}+\mathrm{i}k-\omega)}{\Gamma(\frac{3}{2}+j-\omega)}\mathrm{e}^{(j+1)r} \lim_{r\to\infty} F\left(j+1+\mathrm{i}k,\, j+1-\mathrm{i}k;\, j+\frac{3}{2}-\omega;\, -\sinh^2 r\right) \tag{140}$$

where for simplicity once more we have suppressed the indices a and b of ω. Now we use equation (137) to obtain

$$\lim_{r\to\infty} \Phi^{kj}_{rq}(r) \sim \frac{\Gamma(\frac{1}{2}-\omega+\mathrm{i}k)}{\Gamma(\frac{1}{2}-\omega-\mathrm{i}k)}\frac{\Gamma(-\mathrm{i}k)\Gamma(\frac{1}{2}-\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}\mathrm{e}^{-\mathrm{i}kr} + \frac{\Gamma(\mathrm{i}k)\Gamma(\frac{1}{2}+\mathrm{i}k)}{\Gamma(j+1+\mathrm{i}k)}\mathrm{e}^{\mathrm{i}kr} \tag{141}$$

which is a complex linear combination of incoming and outgoing plane waves. Denoting the common factor of this formula by

$$\Lambda(j, \pm k) \equiv \frac{\Gamma(\pm\mathrm{i}k)\Gamma(\frac{1}{2}\pm\mathrm{i}k)}{\Gamma(j+1\pm\mathrm{i}k)} \tag{142}$$

and the channel dependence factor by

$$\Xi(\omega_{ab}, k) \equiv \frac{\Gamma(\frac{1}{2}-\omega_{ab}+\mathrm{i}k)}{\Gamma(\frac{1}{2}-\omega_{ab}-\mathrm{i}k)} \tag{143}$$

the asymptotic form of our coupled channel wavefunction can be written as

$$\lim_{r\to\infty} \psi^k_{js}(r) \sim \sum_{a,b=-s}^{s} R_a Q_b C\big[\Xi(\omega_{ab},k)\Lambda(j,-k)\mathrm{e}^{-\mathrm{i}kr} + \Lambda(j,k)\mathrm{e}^{\mathrm{i}kr}\big]. \tag{144}$$

For the spin $\frac{1}{2}$ case ω_{ab} is the diagonal matrix diag$(1/2,-1/2)$, hence

$$\Xi\left(\frac{1}{2},k\right) = -\Xi\left(-\frac{1}{2},k\right) = \frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)} \tag{145}$$

due to the relation $\Gamma(1\pm\mathrm{i}k) = \pm\mathrm{i}k\Gamma(\mathrm{i}k)$. Using the explicit form equation (107) of the projectors in equation (144) we get

$$\lim_{r\to\infty} \psi^k_{j\frac{1}{2}}(r) \sim \begin{pmatrix} A_{\frac{1}{2},\frac{1}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} & A_{\frac{1}{2}-\frac{1}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} \\ A_{-\frac{1}{2}\frac{1}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} & A_{-\frac{1}{2}-\frac{1}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} \end{pmatrix} \tag{146}$$

where

$$A_{\frac{1}{2}\frac{1}{2}}(j,k) = A_{-\frac{1}{2}-\frac{1}{2}}(j,k) = \frac{\Gamma(\frac{1}{2}+\mathrm{i}k)}{\Gamma(j+1+\mathrm{i}k)} \tag{147}$$

$$A_{-\frac{1}{2}\frac{1}{2}}(j,k) = A_{\frac{1}{2}-\frac{1}{2}}(j,k) = \frac{\Gamma(\frac{1}{2}-\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}. \tag{148}$$

For spin 1, although the situation is more complicated, the calculations can be carried through in a straightforward manner by using the explicit form, equation (116) of the spin 1 projectors. The result is

$$\lim_{r\to\infty} \psi^k_{j1}(r) \sim \begin{pmatrix} A_{11}(j,k)\mathrm{e}^{\mathrm{i}kr} & 0 & -A_{1-1}(j,k)\mathrm{e}^{-\mathrm{i}kr} \\ 0 & A_{11}(j,k)\mathrm{e}^{\mathrm{i}kr} + \frac{1+\mathrm{i}k}{1-\mathrm{i}k}A_{1-1}(j,k)\mathrm{e}^{-\mathrm{i}kr} & 0 \\ -A_{1-1}(j,k)\mathrm{e}^{-\mathrm{i}kr} & 0 & A_{11}(j,k)\mathrm{e}^{\mathrm{i}kr} \end{pmatrix} \tag{149}$$

where

$$A_{11}(j,k) = A_{-1-1}(j,k) = \frac{\Gamma(\mathrm{i}k)}{\Gamma(j+1+\mathrm{i}k)} \qquad A_{1-1}(j,k) = A_{-11}(j,k) = \frac{\Gamma(-\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}. \tag{150}$$

Notice that in accord with our expectations of section 2 see equations (50) and (51) only the matrix elements satisfying $\lambda^2 = \mu^2$ are different from zero. We have a 2×2 block with $|\lambda| = 1$ and a 1×1 block with $|\lambda| = 0$.

This block structure is also found for the spin $\frac{3}{2}$ case. In this case the result is

$$\lim_{r\to\infty} \psi^k_{j\frac{3}{2}}(r) \sim \begin{pmatrix} A_{\frac{3}{2}\frac{3}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} & 0 & 0 & A_{\frac{3}{2}-\frac{3}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} \\ 0 & A_{\frac{1}{2}\frac{1}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} & A_{\frac{1}{2}-\frac{1}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} & 0 \\ 0 & A_{-\frac{1}{2}\frac{1}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} & A_{-\frac{1}{2}-\frac{1}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} & 0 \\ A_{-\frac{3}{2}\frac{3}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} & 0 & 0 & A_{-\frac{3}{2}-\frac{3}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} \end{pmatrix} \tag{151}$$

where

$$A_{\pm\frac{3}{2}\pm\frac{3}{2}}(j,k) = A_{\pm\frac{1}{2}\pm\frac{1}{2}}(j,k) = \frac{\Gamma(\mathrm{i}k)\Gamma(\frac{1}{2}+\mathrm{i}k)}{\Gamma(j+1+\mathrm{i}k)} \tag{152}$$

$$A_{\pm\frac{3}{2}\mp\frac{3}{2}}(j,k) = \frac{\Gamma(\mathrm{i}k)\Gamma(\frac{1}{2}-\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}\,\frac{\frac{1}{2}-\mathrm{i}k}{\frac{1}{2}+\mathrm{i}k} \tag{153}$$

$$A_{\pm\frac{1}{2}\mp\frac{1}{2}}(j,k) = \frac{\Gamma(\mathrm{i}k)\Gamma(\frac{1}{2}-\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}\,\frac{\frac{3}{2}+\mathrm{i}k}{\frac{3}{2}-\mathrm{i}k}. \tag{154}$$

We consider next how one can extract the physical S matrix from these helicity amplitudes.

8. The scattering matrix

To extract the scattering matrix from the asymptotic form of the coupled channel wavefunction, we start again with the spin $\frac{1}{2}$ case; a case simple enough to gain additional insight for the higher-spin cases. With equation (146) giving the asymptotic form of the coupled channel wavefunction, the matrix can be diagonalized with the help of the transform matrix

$$d^{\frac{1}{2}}(\pi/2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{155}$$

which is just Wigner's d function corresponding to a rotation by an angle $\theta = \frac{\pi}{2}$. The result is

$$d^{\dagger}\psi^{\infty}(r)\,d = \begin{pmatrix} A_{\frac{1}{2}\frac{1}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} + A_{\frac{1}{2}-\frac{1}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} & 0 \\ 0 & A_{\frac{1}{2}\frac{1}{2}}(j,k)\mathrm{e}^{\mathrm{i}kr} - A_{\frac{1}{2}-\frac{1}{2}}(j,k)\mathrm{e}^{-\mathrm{i}kr} \end{pmatrix} \tag{156}$$

where the amplitudes are given by equation (148). From this the scattering phase shifts are readily deduced as

$$\mathrm{e}^{\mathrm{i}\delta_{\pm}} = \pm\frac{\Gamma(\frac{1}{2}-\mathrm{i}k)}{\Gamma(\frac{1}{2}+\mathrm{i}k)}\frac{\Gamma(j+1+\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}. \tag{157}$$

In this base the scattering matrix is diagonal.

But recall that it was in the base of equation (36) that the explicit form of the Casimir operators, wavefunctions and their asymptotics were found. This base can be written alternatively as

$$\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) = \sum_{\lambda'}\chi^{s}_{\lambda'}D^{s}_{\lambda'\lambda}(\varphi,\theta,-\varphi)D^{j}_{\lambda m}(\varphi,-\theta,-\varphi) \tag{158}$$

which contains two D functions transforming according to the corresponding tensor product representation. This tensor product can be reduced by using well known product formulas for D functions [10] to find

$$\mathcal{D}^{js}_{m\lambda}(\theta,\varphi) = \sum_{\lambda'}\chi^{s}_{\lambda'}(-1)^{\lambda-\lambda'}\sum_{l}\langle s-\lambda j\lambda|l0\rangle D^{l}_{0m-\lambda'}(\varphi,-\theta,-\varphi)\langle lm-\lambda'|s-\lambda' jm\rangle. \tag{159}$$

Since $D^{l}_{0m-\lambda'}(\varphi,\theta,-\varphi)$ is just an ordinary spherical harmonic, this expansion resembles a rotated version of a usual expansion of angular momentum states. For $s=\frac{1}{2}$ we have the two values $l = j\pm\frac{1}{2}$ and $-\frac{1}{2}\leqslant\lambda'\leqslant\frac{1}{2}$, and so

$$\begin{aligned}\sqrt{\frac{2j+1}{4\pi}}\mathcal{D}^{j\frac{1}{2}}_{m\lambda} &= \frac{1}{\sqrt{2}}(-1)^{\lambda-\frac{1}{2}}\left(\sqrt{\frac{j-m+1}{2j+2}}Y^{j+\frac{1}{2}}_{m-\frac{1}{2}} - \mathrm{sign}\lambda\sqrt{\frac{j+m}{2j}}Y^{j-\frac{1}{2}}_{m-\frac{1}{2}}\right)\chi^{\frac{1}{2}}_{\frac{1}{2}} \\ &\quad + \frac{1}{\sqrt{2}}(-1)^{\lambda+\frac{1}{2}}\left(\sqrt{\frac{j+m+1}{2j+2}}Y^{j+\frac{1}{2}}_{m+\frac{1}{2}} + \mathrm{sign}\lambda\sqrt{\frac{j-m}{2j}}Y^{j-\frac{1}{2}}_{m+\frac{1}{2}}\right)\chi^{\frac{1}{2}}_{-\frac{1}{2}}.\end{aligned} \tag{160}$$

Here we have used

$$D^{l}_{0m-\lambda'}(\varphi,-\theta,-\varphi) = \sqrt{\frac{4\pi}{2l+1}}Y^{l}_{m-\lambda'}(\theta,\varphi) \tag{161}$$

and the specific values of the Clebsch–Gordan coefficients,

$$\langle 1/2\pm 1/2 j\mp 1/2|j-1/2 0\rangle = \pm\frac{1}{\sqrt{2}} \qquad \langle 1/2\pm 1/2 j\mp 1/2|j+1/2 0\rangle = \frac{1}{\sqrt{2}}. \tag{162}$$

In terms of the usual spinor harmonics $Y_{j\pm\frac{1}{2},j,m}$ [10] our result takes the following form:

$$\sqrt{\frac{2j+1}{4\pi}}\begin{pmatrix}\mathcal{D}^{j\,\frac{1}{2}}_{m\,\frac{1}{2}}\\ \mathcal{D}^{j\,\frac{1}{2}}_{m-\frac{1}{2}}\end{pmatrix}=-\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}Y_{j+\frac{1}{2},j,m}\\ Y_{j-\frac{1}{2},j,m}\end{pmatrix} \tag{163}$$

which clearly shows that the physical basis is just the base provided by the spinor harmonics. Notice also that the matrix of base transformation is given in terms of the Wigner rotation matrix $d^{\frac{1}{2}}(\pi/2)$.

For the spin 1 case we introduce the set of matrices

$$d^1\left(\frac{\pi}{2}\right)=\frac{1}{2}\begin{pmatrix}1&-\sqrt{2}&1\\ \sqrt{2}&0&-\sqrt{2}\\ 1&\sqrt{2}&1\end{pmatrix}\qquad W=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$
$$V=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1&0\\1&1&0\\0&0&\sqrt{2}\end{pmatrix} \tag{164}$$

with which it is straightforward to show that the matrix dWV transforms the 3×3 matrices appearing in the eigenvalue problem of the quadratic Casimir to matrices having the structure of a direct sum of a 2×2 and of a 1×1 one. The corresponding blocks describe $\lambda=\pm 1$ and $\lambda=0$ helicity scattering. The asymptotic form in this new base is

$$(dWV)^\dagger\psi^\infty(dWV)\sim\begin{pmatrix}A_{11}\mathrm{e}^{\mathrm{i}kr}-A_{1-1}\mathrm{e}^{-\mathrm{i}kr}&0&0\\0&A_{11}\mathrm{e}^{\mathrm{i}kr}+\frac{1+\mathrm{i}k}{1-\mathrm{i}k}A_{1-1}\mathrm{e}^{\mathrm{i}kr}&0\\0&0&A_{11}\mathrm{e}^{\mathrm{i}kr}+A_{1-1}\mathrm{e}^{-\mathrm{i}kr}\end{pmatrix} \tag{165}$$

where the amplitudes are given by equation (150). Again it is evident that the eigenphases are given by

$$\mathrm{e}^{\mathrm{i}\delta^j_\pm(k)}=\mp\frac{\Gamma(1-\mathrm{i}k)}{\Gamma(1+\mathrm{i}k)}\frac{\Gamma(j+1+\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}\qquad \mathrm{e}^{\mathrm{i}\delta^j_0(k)}=\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\frac{\Gamma(-\mathrm{i}k)}{\Gamma(\mathrm{i}k)}\frac{\Gamma(j+1+\mathrm{i}k)}{\Gamma(j+1-\mathrm{i}k)}. \tag{166}$$

For the spin $\frac{3}{2}$ case one can proceed similarly. In this case the relevant matrices are

$$d^{\frac{3}{2}}(\pi/2)=\frac{1}{2\sqrt{2}}\begin{pmatrix}1&-\sqrt{3}&\sqrt{3}&-1\\ \sqrt{3}&-1&-1&\sqrt{3}\\ \sqrt{3}&1&1&\sqrt{3}\\ 1&\sqrt{3}&\sqrt{3}&1\end{pmatrix}\qquad W=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \tag{167}$$

and

$$V=\tfrac{1}{2}\begin{pmatrix}1&-\sqrt{3}&0&0\\ \sqrt{3}&1&0&0\\0&0&1&\sqrt{3}\\0&0&-\sqrt{3}&1\end{pmatrix}. \tag{168}$$

This transformation again effects block diagonalization, in this case yielding two 2×2 blocks, with helicities $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$. The transformed asymptotic wavefunction $(dWV)^\dagger\psi^\infty(dWV)$ now has the form

$$\begin{pmatrix}A_{\frac{3}{2}\frac{3}{2}}\mathrm{e}^{\mathrm{i}kr}+A_{\frac{3}{2}-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}kr}&0&0&0\\0&A_{\frac{1}{2}\frac{1}{2}}\mathrm{e}^{\mathrm{i}kr}+A_{\frac{1}{2}-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}kr}&0&0\\0&0&A_{\frac{1}{2}\frac{1}{2}}\mathrm{e}^{\mathrm{i}kr}-A_{\frac{1}{2}-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}kr}&0\\0&0&0&A_{\frac{3}{2}\frac{3}{2}}\mathrm{e}^{\mathrm{i}kr}-A_{\frac{3}{2}-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}kr}\end{pmatrix} \tag{169}$$

where the amplitudes are as given in equations (152)–(154). The phase shifts then result from

$$e^{i\delta^j_{\pm\frac{3}{2}}(k)} = \pm\frac{\Gamma(\frac{3}{2}-ik)}{\Gamma(\frac{3}{2}+ik)}\frac{\Gamma(j+1+ik)}{\Gamma(j+1-ik)} \tag{170}$$

$$e^{i\delta^j_{\pm\frac{1}{2}}(k)} = \pm\frac{\frac{3}{2}+ik}{\frac{3}{2}-ik}\frac{\Gamma(\frac{1}{2}-ik)}{\Gamma(\frac{1}{2}+ik)}\frac{\Gamma(j+1+ik)}{\Gamma(j+1-ik)}. \tag{171}$$

For the arbitrary spin case we can conjecture the general form of the eigenphase shifts of the scattering matrix. To do this, first notice that the eigenphase shifts for the spin $\frac{1}{2}$, 1 and $\frac{3}{2}$ are of the form

$$e^{i\delta^j_\lambda(k)} = \mp(-1)^{2|\lambda|}\frac{\Gamma(|\lambda|-ik)}{\Gamma(|\lambda|+ik)}\frac{\Gamma(j+1+ik)}{\Gamma(j+1-ik)}e^{i\Phi(k)} \qquad j \geqslant s \tag{172}$$

where the meaning of the factor $e^{i\Phi(k)}$ can be clarified as follows. During our calculations we have assumed that $j \geqslant s$. But we also know that the allowed values for j are $|\lambda|, |\lambda|+1, \ldots$. For the spin $\frac{1}{2}$ case this restriction is already satisfied by the choice $j \geqslant s$. But for $s = 1$, we have $|\lambda| = 0, 1$. Hence the allowed values for j are $0, 1, 2, \ldots$ for $|\lambda| = 0$, and $1, 2, \ldots$ for $|\lambda| = 1$. Having the restriction $j \geqslant s$ in this case means that $j \geqslant 1$, which for $|\lambda| = 1$ covers all the cases, but for $|\lambda| = 0$ the case $j = 0$ is missing. It is not hard to show that the $j = 0$ case yields a one-channel scattering problem with the potential $V(r) = \frac{2}{\sinh^2 r}$ from which one obtains the phase shift $-e^{i\Phi(k)} = -\frac{1+ik}{1-ik}$. To within a sign this is precisely the one appearing in the $j \geqslant 1$ case. Hence if we extend the range of j to cover also the $j = 0$ case we will be able to cover all the values $j = 0, 1, 2, \ldots$ allowed for the $\lambda = 0$ case. Specifically

$$e^{i\delta^j_0(k)} = \frac{1+ik}{1-ik}\frac{\Gamma(-ik)}{\Gamma(ik)}\frac{\Gamma(j+1+ik)}{\Gamma(j+1-ik)} \qquad j = 0, 1, \ldots. \tag{173}$$

For the spin $\frac{3}{2}$ case this structure survives as well, for after putting $j = \frac{1}{2}$ in our equations, we get the interaction term

$$V(r) = \frac{2}{\sinh^2 r}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -\cosh r & 0 \\ 0 & -\cosh r & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{174}$$

This 2×2 submatrix is exactly solvable and the result is already known from the spin $\frac{1}{2}$ case (put $j = \frac{3}{2}$ in equation (105) for the potential of the spin $\frac{1}{2}$ case). Hence the eigenphase shift is given by

$$e^{i\delta^{j=\frac{1}{2}}(k)} = \frac{\frac{3}{2}-ik}{\frac{3}{2}+ik}\frac{\frac{1}{2}-ik}{\frac{1}{2}+ik} \tag{175}$$

and we can again extend our result for all j values as follows:

$$e^{i\delta^j_{\pm\frac{1}{2}}(k)} = \pm\frac{\frac{3}{2}+ik}{\frac{3}{2}-ik}\frac{\Gamma(\frac{1}{2}-ik)}{\Gamma(\frac{1}{2}+ik)}\frac{\Gamma(j+1+ik)}{\Gamma(j+1-ik)} \qquad j = \tfrac{1}{2}, \tfrac{3}{2}, \ldots. \tag{176}$$

We have no proof of the conjecture that this adjustment of the phase $e^{i\Phi(k)}$ can consistently be done for all values of the spin, but we surmise

$$e^{i\delta^j_\lambda(k)} = \mp(-1)^{2|\lambda|}e^{i\Phi(k)}\frac{\Gamma(|\lambda|-ik)}{\Gamma(|\lambda|+ik)}\frac{\Gamma(j+1+ik)}{\Gamma(j+1-ik)} \qquad j = |\lambda|, |\lambda|+1, \ldots \tag{177}$$

where $e^{i\Phi(k)}$ is a phase factor needed for the $j \leqslant s$ cases.

We close this section with a few comments on the structure of the asymptotic form of our coupled channel wavefunction as given by equation (144) valid for arbitrary spin s. As we see from this expression the important terms fixing the channel structure are the matrices S and R satisfying equations (89) and (90). It is not hard to prove that the solution of the eigenvalue problem $R\Phi_{\lambda a} = r_a \Phi_{\lambda a}$ is the expression

$$\Phi_{\lambda a} = \frac{1}{\sqrt{(j-\lambda)!(j+\lambda)!(s-\lambda)!(s+\lambda)!}} K_a\left(s+\lambda; \frac{1}{2}, 2s\right) - s \leqslant a \leqslant s \quad - s \leqslant \lambda \leqslant s \tag{178}$$

where $K_a(s+\lambda; \frac{1}{2}, 2s)$ is the Krawtchouk polynomial in $s+\lambda$ of degree a with parameter $\frac{1}{2}$ as discussed in detail in [12]. Using this result the matrices R_a and Q_a can be expressed with these polynomials. Exploiting some well known properties of the Krawtchouk polynomials in the asymptotic formula (144) the phase shifts for arbitrary spin s in principle can be calculated.

Another way of looking at our wavefunction (144) is provided by the observation of section 1 that our generators $\boldsymbol{J}$, $\boldsymbol{M}$ and x^μ provide a realization of the Poincaré algebra. Indeed, these generators are just the ones characterized by the $(1, s)$ representation of the Poincaré group induced by the unitary irreducible representation of the group $SO(3)$ leaving the vector $x^{\star\mu} \equiv (1, 0, 0, 0)$ invariant. Our (74) wavefunction living on the upper sheet of the double-sheeted hyperboloid is just the one transforming according to this induced representation. What we did in the previous sections was just providing this function with labels corresponding to the Lorentz subgroup, $SO(3, 1)$. More precisely we were interested in characterizing the wavefunction with eigenvalues of the $SO(3, 1)$ Casimir operators. This procedure is just the subduction of the $(1, s)$ Poincaré representation to the $SO(3, 1)$ subgroup charcterized by the labels (j_0, j_1). Using this information we should be able to obtain an alternative derivation for the asymptotic form of our wavefunction, and the scattering matrix by purely group theoretical manipulations[3]. These ideas for giving a possible proof for our conjecture will be followed in a subsequent publication.

9. Conclusions

In this paper we have investigated an exactly solvable coupled channel scattering problem with $SO(3, 1)$ symmetry describing the helicity scattering of a particle with spin s. The existence of this exactly solvable problem is based on a special coordinate realization in terms of matrix-valued differential operators. Though the realization is in terms of $(2s+1) \times (2s+1)$ spin matrices, the number of independent channels turns out not to be $2s+1$. Indeed we have shown that the Casimir operators, equations (75) and (76), describe a *collection* of scattering problems where the channels are labelled by the helicity projections $\pm\lambda$ of the spin s. In this picture the scattering problems are only one- or two-channel ones depending on the value of the helicity projection.

We have given a detailed discussion of the group theoretical meaning of the coupled channel wavefunction, showing that this wavefunction is a construct of irreducible unitary representations of the $SO(3, 1)$ algebra. The two different labels of the irreps, k and λ, are related to the scattering energy and the helicity (which gives rise to the channel structure) respectively. We also have shown that the coupled channel wavefunction is a matrix-valued function with definite group theoretical properties. It provides a matrix-valued generalization of known special functions, the matrix-valued hyperbolic Jacobi polinomials in particular.

We have calculated the scattering matrix for the special values $s = \frac{1}{2}, 1, \frac{3}{2}$, and conjectured the result for general s. We demonstrated that for the description of the coupled channel

[3] We are grateful to the referee for drawing our attention to this point.

problem with $SO(3,1)$ symmetry *both* of the independent Casimir operators must be used. One leads to a Schrödinger-like equation describing a collection of helicity scattering problems, while the other behaves like a Dirac-like operator providing the subsidiary conditions amenable for an exact solution. The Schrödinger-like operator, equation (75), and the Dirac-like operator, equation (76), are related to each other similarly to the Hamiltonian (a second-order differential operator) and the supercharge (a first-order differential operator) in supersymmetric quantum mechanics.

As far as AST is concerned these results clearly show that a generalization is needed to describe a coupled channel problem. The first step may be to generalize the theory by allowing more general irreducible unitary representations into the formalism. The extra labels corresponding to the eigenvalues of the extra Casimir operators have to be related somehow to the internal degrees of freedom of the coupled channel scattering process. A further possible step could be to identify the analogue of the so called 'Euclidean connection' [1] for such generalized representations. As was shown and used in a different context in section 3 the Euclidean algebra $e(3)$ also has two independent Casimir operators. Since this algebra arises as a contraction of the $SO(3,1)$ algebra, in principle one should be able to recover our results via a purely algebraic (i.e. realization-independent) method. There is an alternative treatment as given by Kerimov [14] using the intertwining operator method. That approach differs from the spirit of AST, which formulates the theory in terms of group contractions. Nevertheless the functional form of Kerimov's result coincides with that conjectured in equation (177), and is exact for the $s = \frac{1}{2}, 1, \frac{3}{2}$ cases. We also hinted at an approach for establishing a proof for our conjectured functional form of the scattering matrices valid for arbitrary spin. One should also be able to derive this general class of scattering matrices entirely within the framework of AST. Such interesting generalizations we shall address in a subsequent publication.

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Appendix

In this appendix we show that the wavefunction for the spin 1 case is indeed an eigenfunction of $\mathcal{C}_2$. From equation (76), the eigenvalue problem of $\mathcal{C}_2$, the equations to be satisfied in the spin 1 case are

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} + k\right)\psi_{\pm1\pm1}(r) = \frac{\mathrm{i}\alpha}{2\sinh r}\psi_{0\pm1} \qquad \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} - k\right)\psi_{\pm1\mp1}(r) = \frac{\mathrm{i}\alpha}{2\sinh r}\psi_{0\mp1} \tag{179}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\psi_{-10}(r) = \frac{\alpha}{2\sinh r}\psi_{00} \qquad \frac{\mathrm{i}\alpha}{2\sinh r}[\psi_{-1\pm1}(r) - \psi_{1\pm1}(r)] = \pm k\psi_{01}(r) \tag{180}$$

where α is given by equation (112). Moreover, we have the conditions

$$\psi_{10}(r) = \psi_{-10}(r) = \psi_{01}(r) = \psi_{0-1}(r). \tag{181}$$

We check that the explicit wavefunctions given in equations (120)–(122) obtained from the solution of the eigenvalue problem for $\mathcal{C}_1$ do indeed satisfy these equations.

From equations (120)–(122) it is clear that conditions of equation (181) are satisfied. To show that the remaining conditions, equation (179) and equation (180), are satisfied as well, we have to recall some recursion relations for the $\mathcal{B}^l_{mn}(r)$ functions [12]. The relations needed are

$$\frac{j+\frac{1}{2}-\mu\cosh r}{\sinh r}\mathcal{B}_\mu(r)=\frac{1}{2}\left(-\frac{1}{2}-\mu+\mathrm{i}k\right)\mathcal{B}_{\mu+1}(r)-\frac{1}{2}\left(-\frac{1}{2}+\mu+\mathrm{i}k\right)\mathcal{B}_{\mu-1}(r) \qquad (182)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{B}_\mu(r)=\frac{1}{2}\left(-\frac{1}{2}-\mu+\mathrm{i}k\right)\mathcal{B}_{\mu+1}(r)+\frac{1}{2}\left(-\frac{1}{2}+\mu+\mathrm{i}k\right)\mathcal{B}_{\mu-1}(r). \qquad (183)$$

Here we have used the shorthand notation adopted with equation (118). Using these recursion relations a straightforward calculation shows that equations (179) and (180) are really satisfied as we claimed. With this method it is straightforward to check that the wavefunctions obtained for the spin $\frac{1}{2}$ and spin $\frac{3}{2}$ cases also satisfy equation (76).

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